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Computation of Feasible Portfolio Control Strategies for an Insurance Company Using a Discrete Time Asset/Liability Model

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Abstract—A nonlinear discrete time asset/liability model is developed for an insurance company selling investment policies with a guaranteed minimum rate of return and a fixed maturity date. The model accommodates time-dependent investment strategies and transaction costs. At time instants where portfolio rebalancing takes place, the model implements a constraint equation dictating that the total value of assets sold must be equal to the total value of assets purchased plus the total transaction costs. Asset transactions are thus self-financing and no additional cash is required. A procedure is proposed for computing time-dependent portfolio control strategies and the initial shareholders capital, such that given nonlinear financial constraints and requirements are satisfied. Such control strategies are called *feasible portfolio control strategies*.

Keywords—Feasible portfolio control, Dynamic financial analysis, Discrete time asset/liability models, Investment policies with a guaranteed minimum rate of return.

1. INTRODUCTION

This work deals with the discrete time modelling and control of the asset/liability structure of an insurance company. The asset/liability model is defined at discrete time instants $k\Delta$, $k = 0, 1, \dots,$ where Δ is the given basic time period, and time 0 corresponds to the present time.

At time 0, the insurance company obtains funds amounting to L_0 currency units (CU) from sales of investment policies. These policies have a guaranteed minimum rate of return g per basic time period Δ and a specified maturity date $N\Delta$, $N > 0$. That is, after a time period Δ, the

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amount owing to the policyholders is at least $(1+g)L_0$. The insurance company also obtains an initial amount of shareholders funds, $E_0^{(\text{nom})}$ CU. The total capital obtained from policyholders and shareholders, $E_0^{(\text{nom})} + L_0$, is invested in n assets consisting of stocks, bonds, foreign currency deposits, etc., and a risk-free asset. Future predictions of the rates of return of the above-mentioned assets are available. In order to simplify the presentation, the term "asset returns" is used in the sequel and is taken to mean "asset rates of return".

A dynamic nonlinear asset/liability model is developed for the insurance company, extending the basic model given in [1]. The derived model is applied in the following manner. Given the vector of variables describing the asset/liability model at time 0, a predicted trajectory of asset returns, a sequence of portfolio rebalancing time instants, and a sequence of reference portfolio weights, $\mathbf{u}(k) \in \Re^n$, $k = 0, 1, \dots, N$ (a bold font is used to represent vectors and matrices). Then, the dynamic asset/liability model is solved to obtain the asset and liability accounts at time instants $k\Delta$, $k = 0, 1, 2, \dots, N$. The sequence of reference portfolio weights, $\mathbf{u}(0), \dots, \mathbf{u}(N)$, is referred to as a portfolio control strategy and is denoted by $\mathbf{u}$.

At the portfolio rebalancing time instants, asset transactions take place such that the actual portfolio weights equal the precomputed reference portfolio weights. At the rebalancing time instants the asset/liability model implements a constraint equation dictating that the total value of assets sold must be equal to the total value of assets purchased plus the total transaction costs. Asset transactions are thus self-financing and no extra cash is required. At time instants where no portfolio rebalancing takes place, the actual portfolio weights drift according to the predicted trajectory of asset returns.

In this work, the following control problem is considered. Given the above-mentioned asset/liability model, the predicted trajectories of asset returns and a sequence of portfolio rebalancing time instants, then, compute a portfolio control strategy $\mathbf{u}$, and the initial shareholders capital $E_0^{(\text{nom})}$, such that given financial constraints and requirements are satisfied. The requirements include a regulatory constraint that has to be satisfied for all time instants $k\Delta$, $k = 0, 1, \dots, N$, and for all predicted trajectories of asset returns. Additional constraints include lower bounds on risk-adjusted measures of policyholders and shareholders rates of return at time $N\Delta$, and upper and lower bounds on $E_0^{(\text{nom})}$, where the bounds are expressed as fractions of L_0. Note that $E_0^{(\text{nom})}$ is taken as a control parameter in order to satisfy the regulatory constraint.

An important feature of the work presented here is that, instead of solving the above-mentioned problem by some optimal control technique [2], the problem is solved by using the concept of feasible control. The definition of feasible control and the computation of a *feasible portfolio control strategy* is as follows.

FIRST. The portfolio control strategy is parameterized by a finite-dimensional vector $\mathbf{p}$.

SECOND. A penalty function that incorporates all the above-mentioned financial requirements and constraints is constructed. The penalty function is a function of the vector $\mathbf{p}$ and the initial shareholders capital, $E_0^{(\text{nom})}$. The penalty function, $J_0(\mathbf{p}; E_0^{(\text{nom})})$, is constructed in such a manner that it reaches the value of zero if and only if all the requirements and all the constraints are all satisfied.

THIRD. A nonlinear programming algorithm [3] is applied on the vector space in which $(\mathbf{p}; E_0^{(\text{nom})})$ resides to bring the penalty function to zero. In other words, the equation $J_0(\mathbf{p}; E_0^{(\text{nom})}) = 0$ is solved for $(\mathbf{p}; E_0^{(\text{nom})})$, assuming that a solution exists. This yields a solution $(\mathbf{p}^*; E_0^{(\text{nom})*})$ where $\mathbf{p}^*$ parameterizes a *feasible portfolio control strategy*.

Feasible control has been successfully employed in other applications, for example, guidance and control of a supertanker ship in constrained waters [4], closed-loop controller design methods for partially observable linear stochastic systems [5], and control of autonomous vehicles and robotic systems [6].

Some works on asset/liability management (see, for example, [7–10]) use combinations of the following assumptions: linear asset/liability models, linear financial constraints and requirements, no transaction costs, and time-independent portfolio control strategies. To a large extent, this work is an extension of existing methods in asset/liability management. The approach proposed here is more flexible and employs a nonlinear asset/liability model incorporating transaction costs, nonlinear financial constraints and requirements, and a time varying portfolio control strategy.

2. DISCRETE TIME ASSET/LIABILITY MODEL OF AN INSURANCE COMPANY

A discrete time asset/liability model of a hypothetical insurance company is derived in this section.

Denote by $\mathbf{r}(k+1) \in \Re^n$, the vector of predicted asset returns at the end of time interval $(k\Delta, (k+1)\Delta]$, $k = 0, \dots, N-1$. Consider the case where there are ν possible predicted asset return trajectories, $\mathbf{r}(k) \in \Re^n$, $k = 1, \dots, N$. Each such trajectory is called here a scenario. Thus, the following notation is adopted:

$$\mathbf{r}(k; s), \qquad k = 1, \dots, N, \quad s \in S = \{s_1, \dots, s_\nu\}, \tag{1}$$

for a predicted trajectory of asset returns; where $s \in S$ represents a scenario index. Note that asset 1 to asset $n-1$ are risky assets while asset n is assumed to be a risk-free asset. The predicted returns have to satisfy the constraints, $-1 < r_i(k; s) < C$, $i = 1, \dots, n-1$, $0 < r_n(k; s) < C$, for all $k = 1, \dots, N$, and for all $s \in S$, where C is some finite positive number. More details on the generation of predicted returns[1] for the risky assets are given in Appendix 2.

The predicted rate of return for the risk-free asset is given by,

$$r_n(k; s) = r_f, \qquad k = 1, \dots, N, \quad \forall\, s \in S, \tag{2}$$

where $r_f > 0$ is a specified constant. The risk-free asset is assumed here to be a bank cash deposit account.

A nonnegative weight p_s is assigned to each scenario $s \in S$ such that

$$p_s \ge 0, \qquad \forall\, s \in S, \quad \sum_{s \in S} p_s = 1. \tag{3}$$

The balance sheet of the insurance company consists of asset accounts and liability accounts (see Figure 1).

Assets	Liabilities
1. Investment 1: $X_1(k; s)$	1. Policyholders Liability: $L(k; s)$
2. Investment 2: $X_2(k; s)$	2. Nominal Equity: $E^{(\mathrm{nom})}(k; s)$
3. Investment 3: $X_3(k; s)$	3. Equity Reserve: $E^{(\mathrm{res})}(k; s)$
⋮	
⋮	
n. Investment n (cash): $X_n(k; s)$	
Total Assets: $A(k; s)$	Total Liabilities: $L(k; s) + E^{(\mathrm{nom})}(k; s) + E^{(\mathrm{res})}(k; s)$

Figure 1. Schematic of the insurance company balance sheet at time instant $k\Delta$, $k = 0, 1, \dots, \forall\, s \in S$.

[1]The return or rate of return of asset i, $i = 1, \dots, n-1$, at the end of time interval $(k\Delta, (k+1)\Delta]$, is defined here by

$$\frac{P_i(k+1) - P_i(k)}{P_i(k)}, \qquad i = 1, \dots, n-1, \quad k = 0, 1, \dots, N-1,$$

where $P_i(k)$ is the price of asset i at time instant $k\Delta$, $k = 0, 1, \dots, N$, $i = 1, \dots, n-1$.

The company assets consist of investments in stocks, bonds, etc., and a bank cash deposit account. The market value of the investment in asset i, at time instant $k\Delta$, is denoted by $X_i(k;s) \geq 0$ and is recorded in asset account i, $i = 1, \ldots, n$, $k = 0, 1, \ldots, N$, $\forall\, s \in S$. The sum total of all the investments at time instant $k\Delta$ is denoted by $A(k; s)$,

$$A(k; s) = \sum_{i=1}^{n} X_i(k; s) > 0, \qquad k = 0, 1, \ldots, N, \quad \forall\, s \in S. \tag{4}$$

The company liabilities consist of the liability to all the policyholders, $L(k; s)$, the nominal equity or nominal liability to all the shareholders, $E^{(\mathrm{nom})}(k; s)$, and the equity reserve, $E^{(\mathrm{res})}(k; s)$, $k = 0, 1, \ldots, N$, $\forall\, s \in S$.

The constraint equation that is satisfied at all times is given by (see [11])

$$A(k; s) = L(k; s) + E^{(\mathrm{nom})}(k; s) + E^{(\mathrm{res})}(k; s), \qquad k = 0, 1, \ldots, N, \quad \forall\, s \in S. \tag{5}$$

Thus, given $A(k; s)$, $L(k; s)$, and $E^{(\mathrm{nom})}(k; s)$, the equity reserve is computed using (5),

$$E^{(\mathrm{res})}(k; s) = A(k; s) - L(k; s) - E^{(\mathrm{nom})}(k; s), \qquad k = 0, 1, \ldots, N, \quad \forall\, s \in S. \tag{6}$$

The initial values of the liability accounts are as follows: $L(0; s) = L_0 > 0$, $\forall\, s \in S$, is the total capital received from policyholders, $E^{(\mathrm{res})}(0; s) = 0$, $\forall\, s \in S$, and $E^{(\mathrm{nom})}(0; s) = E_0^{(\mathrm{nom})} > 0$, $\forall\, s \in S$, is the initial shareholders capital.

Thus, the initial total capital available for investment is given by

$$A(0; s) = A_0 = L_0 + E_0^{(\mathrm{nom})} > 0, \qquad \forall\, s \in S. \tag{7}$$

Given L_0, $E_0^{(\mathrm{nom})}$ has to be chosen such that the following regulatory constraint (discussed later) is satisfied:

$$\frac{A_0 - L_0}{L_0} \geq \rho \Rightarrow \frac{E_0^{(\mathrm{nom})}}{L_0} \geq \rho, \tag{8}$$

where ρ is a specified constant, $0 < \rho < 1$.

The weight of investment i in the overall portfolio, $w_i(k; s)$, is defined here by

$$w_i(k; s) = \frac{X_i(k; s)}{A(k; s)}, \qquad i = 1, \ldots, n, \quad k = 0, 1, \ldots, N, \quad \forall\, s \in S, \tag{9}$$

and

$$0 \leq w_i(k; s) \leq 1, \quad i = 1, \ldots, n, \qquad \sum_{i=1}^{n} w_i(k; s) = 1, \quad k = 0, 1, \ldots, N, \quad \forall\, s \in S. \tag{10}$$

The rate of return of the investment portfolio at the end of time interval $(k\Delta, (k+1)\Delta]$ is given by

$$R^{(p)}(k+1; s) = \sum_{i=1}^{n} w_i(k; s) r_i(k+1; s), \qquad k = 0, 1, \ldots, N-1, \quad \forall\, s \in S. \tag{11}$$

Let $\mathbf{u}(k) \in \Re^n$, $k = 0, 1, \ldots, N$, be a given sequence of reference portfolio weights, satisfying,

$$0 \leq u_i(k) \leq 1, \quad i = 1, \ldots, n, \qquad \sum_{i=1}^{n} u_i(k) = 1, \quad k = 0, 1, \ldots, N-1, \tag{12}$$

and where it is assumed that $\mathbf{u}(N) = \mathbf{u}(N-1)$. The sequence of reference portfolio weights is referred to as a portfolio control strategy, and is denoted by $\mathbf{u}$.

It is assumed that the initial portfolio weights vector $\mathbf{w}(0; s)$ satisfies,

$$\mathbf{w}(0; s) = \mathbf{u}(0), \qquad \forall\, s \in S. \tag{13}$$

Let 0^- denote a time instant just before time 0. At time 0^-, the total company assets consist of cash obtained from policyholders and shareholders. This implies that $\mathbf{w}(0^-; s) = [0, 0, \ldots, 0, 1]^\top$, $\forall\, s \in S$. At a time instant between 0^- and 0, the cash is invested in n assets such that at time 0, (13) is satisfied. The transaction costs involved in this particular transaction are not accounted for in the dynamic model. Thus, $X_i(0^-; s) = 0$, $i = 1, \ldots, n-1$, $X_n(0^-; s) = A_0$, $A(0^-; s) = A_0$, $X_i(0; s) = u_i(0)A_0$, $i = 1, \ldots, n$, $A(0; s) = A_0$, $\forall s \in S$.

The asset/liability model consists of a set of difference equations. Each difference equation can be used to calculate the value of a particular asset/liability account at time instant $(k+1)\Delta$, given the value of the account and the values of some other variables at time instant $k\Delta$, $k = 0, 1, \ldots$, and, given a specific scenario $s \in S$, and a portfolio control strategy $\mathbf{u}$.

The difference equation for the policyholders liability account, $L(k; s)$, is given by

$$\begin{gathered} L(k+1; s) = [1 - \lambda(k+1)]L(k; s) \left(1 + \max\left[\theta R^{(p)}(k+1; s), g\right]\right), \qquad L(0; s) = L_0, \\ \forall\, s \in S, \qquad k = 0, 1, \ldots, N-1, \end{gathered} \tag{14}$$

where $0 \le \lambda(k+1) \ll 1$, $k = 0, 1, \ldots, N-1$, is a given time-dependent function representing the fraction of policyholders liability that is paid out at the end of each time interval $(k\Delta, (k+1)\Delta]$, $k = 0, 1, \ldots, N-1$, due to investment policies being surrendered. In addition, $0 < \theta < 1$ is a specified fraction of the portfolio rate of return apportioned to policyholders.

From (14), the cash amount, $\zeta_1(k+1; s)$, paid out to policyholders surrendering their policies, is given by

$$\begin{gathered} \zeta_1(k+1; s) = \lambda(k+1)L(k; s) \left(1 + \max\left[\theta R^{(p)}(k+1; s), g\right]\right), \qquad k = 0, 1, \ldots, N-1, \\ \forall\, s \in S. \end{gathered} \tag{15}$$

This cash amount is withdrawn from the bank cash deposit account (X_n).

If $\theta R^{(p)}(k+1; s) < g$, then there is a shortfall amount, $\zeta_2(k+1; s)$, given by

$$\zeta_2(k+1; s) = L(k; s) \max\left[0, g - \theta R^{(p)}(k+1; s)\right], \qquad k = 0, 1, \ldots, N-1, \quad \forall\, s \in S. \tag{16}$$

In this work, it is assumed that if $\theta R^{(p)}(k+1; s) < g$, then additional company shares are sold, with total value equal to the shortfall amount ζ_2. The cash obtained is deposited in the bank cash deposit account (X_n) while the cash figure is added to the nominal equity account.

The difference equation for the nominal equity, $E^{(\text{nom})}(k; s)$, is given by

$$\begin{gathered} E^{(\text{nom})}(k+1; s) = E^{(\text{nom})}(k; s) + L(k; s) \max\left[0, g - \theta R^{(p)}(k+1; s)\right], \\ E^{(\text{nom})}(0; s) = E_0^{(\text{nom})}, \qquad \forall\, s \in S, \quad k = 0, 1, \ldots, N-1. \end{gathered} \tag{17}$$

In this work, the total rate of return on shareholders capital at the end of time interval $(0, k\Delta]$, $y^{(\text{sh})}(k; s)$, is defined by

$$y^{(\text{sh})}(k; s) = \frac{A(k; s) - L(k; s)}{E^{(\text{nom})}(k; s)}, \qquad k = 0, 1, \ldots, N, \quad \forall\, s \in S. \tag{18}$$

The policyholders liability for the case where there are no policy surrenders during the time interval $[0, N\Delta]$ is denoted by $L^{(\text{nos})}(k; s)$, and the difference equation is given by

$$\begin{gathered} L^{(\text{nos})}(k+1; s) = L^{(\text{nos})}(k; s) \left(1 + \max\left[\theta R^{(p)}(k+1; s), g\right]\right), \qquad L^{(\text{nos})}(0; s) = L_0, \\ \forall\, s \in S, \quad k = 0, 1, \ldots, N-1. \end{gathered} \tag{19}$$

In this work, the total rate of return on policyholders capital at the end of time interval $(0, k\Delta]$, $y^{(\mathrm{pol})}(k;s)$, is defined by

$$y^{(\mathrm{pol})}(k;s) = \frac{L^{(\mathrm{nos})}(k;s)}{L_0}, \qquad k = 0,\dots,N, \quad \forall\, s \in S. \tag{20}$$

Let $(k+1)^{-}\Delta$ denote the time instant just before $(k+1)\Delta$, $k = 0,1,\dots$. The total investments at time instant $(k+1)^{-}\Delta$, $A((k+1)^{-};s)$, is given by

$$\begin{gathered} A\left((k+1)^{-};s\right) = \left(1 + R^{(p)}(k+1;s)\right) A(k;s) - \zeta_1(k+1;s) + \zeta_2(k+1;s), \\ k = 0,1,\dots,N-1, \qquad \forall\, s \in S. \end{gathered} \tag{21}$$

The market value of investment i, $i = 1,\dots,n-1$, at time instant $(k+1)^{-}\Delta$, $X_i((k+1)^{-};s)$, is given by

$$\begin{aligned} X_i\left((k+1)^{-};s\right) &= (1 + r_i(k+1;s))X_i(k;s), \\ &= (1 + r_i(k+1;s))w_i(k;s)A(k;s), \\ k &= 0,1,\dots,N-1, \quad \forall\, s \in S, \\ i &= 1,\dots,n-1. \end{aligned} \tag{22}$$

The amount in the bank cash deposit account at time instant $(k+1)^{-}\Delta$, $X_n((k+1)^{-};s)$, is given by

$$\begin{aligned} X_n\left((k+1)^{-};s\right) &= (1 + r_f)X_n(k;s) - \zeta_1(k+1;s) + \zeta_2(k+1;s), \\ &= (1 + r_f)w_n(k;s)A(k;s) - \zeta_1(k+1;s) + \zeta_2(k+1;s), \\ k &= 0,1,\dots,N-1, \qquad \forall\, s \in S. \end{aligned} \tag{23}$$

In order to simplify the notation in (21)–(23), it has been assumed that $\zeta_1(k+1;s) = \zeta_1((k+1)^{-};s)$, $\zeta_2(k+1;s) = \zeta_2((k+1)^{-};s)$, $r_i(k+1;s) = r_i((k+1)^{-};s)$, $i = 1,\dots,n$, $R^{(p)}(k+1;s) = R^{(p)}((k+1)^{-};s)$, $k = 0,1,\dots,N-1$, $\forall\, s \in S$. Also, note from (21),(23) that it has been assumed that withdrawals from and deposits into the bank cash deposit account (X_n) incur no transaction costs.

Since the amount ζ_1 is subtracted from the cash deposit account, (23), and thus also from total assets, (21), the following constraints are imposed here (see later): $X_i((k+1)^{-};s) \geq 0$, $i = 1,\dots,n$, and $A((k+1)^{-};s) > 0$, for all $k = 0,\dots,N-1$, and for all $s \in S$.

Let T_B, T_Q be predefined sets of time indices satisfying

$$T_B \subseteq Z_0 = \{1,2,\dots,N\}, \qquad T_Q = Z_0 - T_B. \tag{24}$$

If portfolio rebalancing is to take place just before time instant $(k+1)\Delta$, then $(k+1) \in T_B$, otherwise $(k+1) \in T_Q$, $k = 0,1,\dots,N-1$. It is assumed here that the rebalancing time instants divide the time horizon $[0, N\Delta]$ into equal consequtive time intervals. Difference equations for the portfolio weights and total investments are developed next for the case where portfolio rebalancing takes place and for the case where no portfolio rebalancing takes place.

CASE 1. PORTFOLIO REBALANCING. $k \in \{0,1,\dots,N-1\}$ and $(k+1) \in T_B$.

In this case, at a time instant between $(k+1)^{-}\Delta$ and $(k+1)\Delta$, the investment portfolio is rebalanced such that the following is satisfied:

$$w_i(k+1;s) = u_i(k+1), \qquad \forall\, s \in S, \quad i = 1,\dots,n. \tag{25}$$

The value of investment i after rebalancing is given by

$$X_i(k+1;s) = u_i(k+1)A(k+1;s), \qquad \forall\, s \in S, \tag{26}$$

where the total investments after rebalancing, $A(k+1;s)$, is determined below. The transaction cost for rebalancing investment i, $\gamma_i(k+1;s)$, is defined here by

$$\begin{aligned}\gamma_i(k+1;s) &= b_i \left|X_i(k+1;s) - X_i\left((k+1)^-;s\right)\right|, \\ &= b_i \left|u_i(k+1)A(k+1;s) - X_i\left((k+1)^-;s\right)\right|, \\ &\qquad \forall\, s \in S, \quad i = 1,\dots,n,\end{aligned} \tag{27}$$

where $0 \le b_i \ll 1$, $i = 1,\dots,n$.

The following constraint equation is assumed here:

$$\begin{aligned}A(k+1;s) &= A\left((k+1)^-;s\right) - \sum_{i=1}^{n} \gamma_i(k+1;s), \\ &= A\left((k+1)^-;s\right) - \sum_{i=1}^{n} b_i \left|u_i(k+1)A(k+1;s) - X_i\left((k+1)^-;s\right)\right|, \\ &\qquad \forall\, s \in S,\end{aligned} \tag{28}$$

where $A((k+1)^-;s)$, $\forall\, s \in S$, is obtained from (21). Equation (28) has to be solved for $A(k+1;s)$ subject to the constraint (see (4))

$$A(k+1;s) > 0, \qquad \forall\, s \in S. \tag{29}$$

If (28),(29) have a solution, then (28) can be rewritten in the following manner,

$$\begin{aligned}&\sum_{i\in\mathcal{D}_1(k+1;s)} \left(X_i\left((k+1)^-;s\right) - X_i(k+1;s)\right) = \\ &\sum_{i\in\mathcal{D}_2(k+1;s)} \left(X_i((k+1);s) - X_i\left((k+1)^-;s\right)\right) \\ &+ \sum_{i=1}^{n} b_i \left|X_i(k+1;s) - X_i\left((k+1)^-;s\right)\right|, \qquad \forall\, s \in S,\end{aligned} \tag{30}$$

where

$$\mathcal{D}_1(k+1;s) = \left\{i \in \{1,\dots,n\} \mid X_i\left((k+1)^-;s\right) - X_i(k+1;s) \ge 0\right\}, \qquad \forall\, s \in S, \tag{31}$$

$$\mathcal{D}_2(k+1;s) = \left\{i \in \{1,\dots,n\} \mid X_i\left((k+1)^-;s\right) - X_i(k+1;s) < 0\right\}, \qquad \forall\, s \in S. \tag{32}$$

$\mathcal{D}_1(k+1;s)$ is the set of indices of assets which were sold or kept constant while $\mathcal{D}_2(k+1;s)$ is the set of indices of assets which were purchased, $\forall\, s \in S$. Equation (30) implies that the total value of assets sold equals exactly the total value of assets purchased plus the selling and buying transaction costs. Asset transactions are thus self-financing and no extra cash is required.

CASE 2. NO PORTFOLIO REBALANCING. $k \in \{0,1,\dots,N-1\}$ and $(k+1) \in T_Q$.

In this case, no portfolio rebalancing takes place, implying that

$$X_i(k+1;s) = X_i\left((k+1)^-;s\right), \qquad \forall\, s \in S, \quad i = 1,\dots,n. \tag{33}$$

The difference equations for the total investments and the portfolio weights are

$$A(k+1;s) = A\left((k+1)^-;s\right), \qquad \forall\, s \in S, \tag{34}$$

$$w_i(k+1;s) = \frac{X_i\left((k+1)^-;s\right)}{A\left((k+1)^-;s\right)}, \qquad \forall\, s \in S, \quad i = 1,\dots,n. \tag{35}$$

From (35),(22),(23), it follows that the portfolio weights drift according to the predicted trajectories of asset returns. Note that if $T_B = \emptyset$ then no rebalancing takes place, while if $T_B = \{1, 2, \ldots, N\}$ then rebalancing takes place at every time instant $k\Delta$, $k = 1, 2, \ldots, N$.

Thus, there are a total of $m = 4 + n$ variables that describe the asset/liability model. These variables can be grouped into a vector $\mathbf{x}(k; s) \in \Re^m$, as follows:

$$\begin{gathered}\mathbf{x}(k; s) = \left[L(k; s), E^{(\text{nom})}(k; s), L^{(\text{nos})}(k; s), A(k; s), w_1(k; s), \ldots, w_n(k; s)\right]^\top, \\ k = 0, 1, \ldots, N, \quad \forall\, s \in S.\end{gathered} \tag{36}$$

Given L_0, $E_0^{(\text{nom})}$ and $\mathbf{u}(0) \in \Re^n$, the initial values of the variables are computed from (7),(13),

$$\mathbf{x}(0; s) = \mathbf{x}_0 = \left[L_0, E_0^{(\text{nom})}, L_0, L_0 + E_0^{(\text{nom})}, \mathbf{u}^\top(0)\right]^\top, \qquad \forall\, s \in S. \tag{37}$$

The dynamic model is summarized below for convenience:

$$\begin{gathered}L(k+1; s) = [1 - \lambda(k+1)]L(k; s)\left(1 + \max\left[\theta R^{(p)}(k+1; s), g\right]\right), \\ k = 0, 1, \ldots, N-1, \qquad L(0; s) = L_0, \quad \forall\, s \in S,\end{gathered} \tag{38}$$

$$\begin{gathered}E^{(\text{nom})}(k+1; s) = E^{(\text{nom})}(k; s) + L(k; s) \max\left[0, g - \theta R^{(p)}(k+1; s)\right], \\ k = 0, 1, \ldots, N-1, \qquad E^{(\text{nom})}(0; s) = E_0^{(\text{nom})}, \quad \forall\, s \in S,\end{gathered} \tag{39}$$

$$\begin{gathered}L^{(\text{nos})}(k+1; s) = L^{(\text{nos})}(k; s)\left(1 + \max\left[\theta R^{(p)}(k+1; s), g\right]\right), \\ k = 0, 1, \ldots, N-1, \qquad L^{(\text{nos})}(0; s) = L_0, \quad \forall\, s \in S,\end{gathered} \tag{40}$$

$$A(0; s) = L_0 + E_0^{(\text{nom})}, \qquad \forall\, s \in S, \quad \mathbf{w}(0; s) = \mathbf{u}(0), \quad \forall\, s \in S, \tag{41}$$

$$R^{(p)}(k+1; s) = \sum_{i=1}^{n} w_i(k; s) r_i(k+1; s), \qquad k = 0, 1, \ldots, N-1, \quad \forall\, s \in S, \tag{42}$$

$$\begin{gathered}\zeta_1(k+1; s) = \lambda(k+1)L(k; s)\left(1 + \max\left[\theta R^{(p)}(k+1; s), g\right]\right), \qquad k = 0, 1, \ldots, N-1, \\ \forall\, s \in S,\end{gathered} \tag{43}$$

$$\zeta_2(k+1; s) = L(k; s) \max\left[0, g - \theta R^{(p)}(k+1; s)\right], \qquad k = 0, 1, \ldots, N-1, \quad \forall\, s \in S, \tag{44}$$

$$\begin{gathered}A\left((k+1)^-; s\right) = \left(1 + R^{(p)}(k+1; s)\right) A(k; s) - \zeta_1(k+1; s) + \zeta_2(k+1; s), \\ k = 0, 1, \ldots, N-1, \qquad \forall\, s \in S,\end{gathered} \tag{45}$$

$$\begin{gathered}X_i\left((k+1)^-; s\right) = (1 + r_i(k+1; s))w_i(k; s)A(k; s), \qquad k = 0, 1, \ldots, N-1; \\ \forall\, s \in S, \qquad i = 1, \ldots, n-1,\end{gathered} \tag{46}$$

$$\begin{gathered}X_n\left((k+1)^-; s\right) = (1 + r_f)w_n(k; s)A(k; s) - \zeta_1(k+1; s) + \zeta_2(k+1; s), \\ k = 0, 1, \ldots, N-1, \qquad \forall\, s \in S.\end{gathered} \tag{47}$$

CASE 1. PORTFOLIO REBALANCING: $k \in \{0, 1, \ldots, N-1\}$ AND $(k+1) \in T_B$.

$$w_i(k+1; s) = u_i(k+1), \qquad \forall\, s \in S, \quad i = 1, \ldots, n, \tag{48}$$

$$A(k+1; s) = A\left((k+1)^-; s\right) - \sum_{i=1}^{n} b_i \left|u_i(k+1)A(k+1; s) - X_i\left((k+1)^-; s\right)\right|, \quad \forall\, s \in S, \tag{49}$$

subject to the constraint

$$A(k+1;s) > 0, \qquad \forall\, s \in S. \tag{50}$$

CASE 2. NO PORTFOLIO REBALANCING: $k \in \{0, 1, \dots, N-1\}$ AND $(k+1) \in T_Q$.

$$A(k+1;s) = A\left((k+1)^-;s\right), \qquad \forall\, s \in S, \tag{51}$$

$$w_i(k+1;s) = \frac{X_i\left((k+1)^-;s\right)}{A\left((k+1)^-;s\right)}, \qquad \forall\, s \in S, \quad i = 1, \dots, n. \tag{52}$$

Note that the model parameters,

$$T_B, \quad L_0, \quad E_0^{(\mathrm{nom})}, \quad r_f, \quad g, \quad \theta, \quad \lambda(k), \qquad k = 1, \dots, N, \tag{53}$$

and predicted trajectories of asset returns, (1), have to be specified. In this work, it is assumed that the guaranteed minimum rate of return g is chosen to be a little less than the risk-free rate of return r_f.

Given an initial condition $\mathbf{x}_0$, (37), a scenario $s \in S$, a set of parameters (53), and a portfolio control strategy, $\mathbf{u}(k) \in \Re^n$, $k = 0, 1, \dots, N$. Then, the discrete time asset/liability model, (38)–(52), is solved to obtain $\mathbf{x}(k;s)$, $k = 0, 1, \dots, N$, (36).

For later use, the center of mass of a function ϕ at time instant $k\Delta$, $M(\phi;k)$, is defined by

$$M(\phi;k) = \sum_{s \in S} p_s \phi(k;s), \qquad k = 0, 1, \dots, \tag{54}$$

where $\phi : \{0, 1, \dots\} \times S \to \Re^{m_0}$ is a bounded continuous function.

The second moment about the center of mass, $V(\phi;k)$, is given by

$$V(\phi;k) = \sum_{s \in S} p_s (\phi(k;s) - M(\phi;k))(\phi(k;s) - M(\phi;k))^\top, \qquad k = 0, 1, \dots. \tag{55}$$

The function D is defined for the case $m_0 = 1$, and is given by

$$D(\phi;k) = [V(\phi;k)]^{1/2}, \qquad k = 0, 1, \dots. \tag{56}$$

3. FORMULATION OF THE FEASIBLE PORTFOLIO CONTROL PROBLEM

The insurance company has to compute $E_0^{(\mathrm{nom})}$ and a portfolio control strategy $\mathbf{u}(k) \in \Re^n$, $k = 0, \dots, N$, for the discrete time asset/liability model, (38)–(52), such that the following constraints and requirements are satisfied:

$$M\left(y^{(\mathrm{sh})};N\right) - d^{(\mathrm{sh})} D\left(y^{(\mathrm{sh})};N\right) \geq e^{(\mathrm{sh})}, \tag{57}$$

$$M\left(y^{(\mathrm{pol})};N\right) - d^{(\mathrm{pol})} D\left(y^{(\mathrm{pol})};N\right) \geq e^{(\mathrm{pol})}, \tag{58}$$

where $y^{(\mathrm{sh})}$ and $y^{(\mathrm{pol})}$ are given by (18) and (20), respectively, $d^{(i)}$, $e^{(i)} > 0$, $i = \mathrm{sh}, \mathrm{pol}$, are given numbers, and

$$\begin{aligned}
X_i(k^-;s) &\geq 0, \quad i = 1, \dots, n, \qquad A(k^-;s) \geq d_0, \quad k = 0, 1, \dots, N, \quad \forall\, s \in S, \\
X_i(k;s) &\geq 0, \quad i = 1, \dots, n, \qquad A(k;s) \geq d_0, \quad k = 0, 1, \dots, N, \quad \forall\, s \in S, \\
&\frac{A(k;s) - L(k;s)}{L(k;s)} \geq \rho, \qquad k = 0, 1, \dots, N, \quad \forall\, s \in S,
\end{aligned} \tag{59}$$

where $0 < d_0 < A_0$, ρ is a given number such that, $0 < \rho < 1$, and

$$\rho L_0 \le E_0^{(\text{nom})} \le \rho_1 L_0, \tag{60}$$

where ρ_1 is a given number such that, $\rho < \rho_1 < 1$, and

$$0 \le u_i(k) \le 1, \qquad i = 1, \dots, n, \quad k = 0, 1, \dots, N, \tag{61}$$

$$\sum_{i=1}^{n} u_i(k) = 1, \qquad k = 0, 1, \dots, N. \tag{62}$$

If it is assumed that $D(y^{(i)}; \cdot)$, $i = \text{sh}, \text{pol}$, is a measure of riskiness of the total rate of return $y^{(i)}$, $i = \text{sh}, \text{pol}$, then constraints (57),(58) represent lower bounds on risk-adjusted center of mass values of $y^{(\text{sh})}$ and $y^{(\text{pol})}$, respectively. In this manner, both the interests of shareholders and policyholders are taken into account.

The first two parts of (59) imply that all the investment accounts must be nonnegative, and that the total assets must be greater than or equal to d_0 for all time instants and for all scenarios $s \in S$, thus limiting losses. The last part of (59) represents a regulatory constraint specifying that the ratio of accounting equity to policyholders liability has to be greater than ρ, for all time instants and for all scenarios $s \in S$. Constraints (60) represent upper and lower bounds on the initial shareholders capital. Constraints (61),(62) are control constraints, (12).

A portfolio control strategy satisfying constraints (57)–(62), will be called here a *feasible portfolio control strategy*. In order to compute a feasible portfolio control strategy, the following procedure is applied.

FIRST. The portfolio control strategy is constructed in the following manner. Consider a sequence of vectors $\mathbf{p}_i = [p_{i1}, p_{i2}, \dots, p_{in}]^\top \in \Re^n \setminus \{\mathbf{0}\}$, $i = 1, \dots, v_0$, where v_0 is a specified integer such that $1 \le v_0 \le \min(N, a_0)$, $a_0 = 1+$ number of rebalancing time indices in the set T_B. Let the vectors $\boldsymbol{\pi}_i = [\pi_{i1}, \pi_{i2}, \dots, \pi_{in}]^\top$, $i = 1, \dots, v_0$, be given by

$$\pi_{ij} = \frac{p_{ij}^2}{\sum\limits_{j=1}^{n} p_{ij}^2}, \qquad j = 1, \dots, n, \quad i = 1, \dots, v_0. \tag{63}$$

Thus,

$$0 \le \pi_{ij} \le 1, \quad j = 1, \dots, n, \quad i = 1, \dots, v_0, \qquad \sum_{j=1}^{n} \pi_{ij} = 1, \quad i = 1, \dots, v_0. \tag{64}$$

Let

$$\mathbf{u}(k) = \begin{cases} \boldsymbol{\pi}_1, & \text{if } 0 \le k < k_0, \\ \boldsymbol{\pi}_2, & \text{if } k_0 \le k < 2k_0, \\ \vdots & \vdots \\ \boldsymbol{\pi}_{v_0}, & \text{if } (v_0 - 1)k_0 \le k < \min(N, v_0 k_0), \end{cases} \tag{65}$$

where

$$k_0 = \begin{cases} \text{int}\left(\dfrac{N}{v_0}\right) + 1, & \text{if } \dfrac{N}{v_0} \text{ is not an integer}, \\ \dfrac{N}{v_0}, & \text{if } \dfrac{N}{v_0} \text{ is an integer}, \end{cases} \tag{66}$$

$\text{int}(\chi)$ is the integral part of χ, and it is assumed that $\mathbf{u}(N) = \mathbf{u}(N-1)$. It then follows that the portfolio control strategy $\mathbf{u}$ satisfies the control constraints (61),(62).

SECOND. Define the following penalty function:

$$\begin{aligned}
J_0\left(\mathbf{p}; E_0^{(\text{nom})}\right) &= \alpha_1 G_1\left(M\left(y^{(\text{sh})};N\right) - d^{(\text{sh})} D\left(y^{(\text{sh})};N\right), e^{(\text{sh})}, B\right) \\
&+\alpha_2 G_1\left(M\left(y^{(\text{pol})};N\right) - d^{(\text{pol})} D\left(y^{(\text{pol})};N\right), e^{(\text{pol})}, B\right) \\
&+\alpha_3 \sum_{s\in S}\sum_{k=0}^{N}\sum_{i=1}^{n} G_1\left(X_i\left(k^-;s\right),0,B\right) + \alpha_3 \sum_{s\in S}\sum_{k=0}^{N}\sum_{i=1}^{n} G_1\left(X_i\left(k;s\right),0,B\right) \\
&+\alpha_3 \sum_{s\in S}\sum_{k=0}^{N} G_1\left(A\left(k^-;s\right),d_0,B\right) + \alpha_3 \sum_{s\in S}\sum_{k=0}^{N} G_1\left(A\left(k;s\right),d_0,B\right) \\
&+\alpha_3 \sum_{s\in S}\sum_{k=0}^{N} G_1\left(\frac{A(k;s)-L(k;s)}{L(k;s)},\rho,B\right) + \alpha_4 G_1\left(E_0^{(\text{nom})},\rho L_0,\rho_1 L_0\right),
\end{aligned} \tag{67}$$

where

$$\mathbf{p} = \left[\mathbf{p}_1^{\mathsf{T}},\dots,\mathbf{p}_{v_0}^{\mathsf{T}}\right]^{\mathsf{T}}, \tag{68}$$

B is a big positive number, $\alpha_i > 0$, $i = 1,\dots,4$, are given numbers, and where

$$G_1(\zeta, c_1, c_2) = \begin{cases} (\zeta - c_2)^2, & \text{if } \zeta > c_2, \\ 0, & \text{if } c_1 \le \zeta \le c_2, \\ (\zeta - c_1)^2, & \text{if } \zeta < c_1, \end{cases} \tag{69}$$

$\zeta, c_1, c_2 \in \Re$, $c_1 < c_2$.

THIRD. Compute a solution $\mathbf{p}^*$ and $E_0^{(\text{nom})*}$ to the following equation:

$$J_0\left(\mathbf{p}^*; E_0^{(\text{nom})*}\right) = 0, \tag{70}$$

assuming a solution exists. Clearly, if there is a vector $\mathbf{p}^*$ and number $E_0^{(\text{nom})*}$ such that $J_0(\mathbf{p}^*; E^{(\text{nom})*}) = 0$, then all the constraints and requirements are satisfied, and the corresponding control strategy, denoted by $\mathbf{u}^*$, is a *feasible portfolio control strategy*.

Practically, the computation of $(\mathbf{p}^*; E_0^{(\text{nom})*})$ is conducted by solving an unconstrained minimization problem on $\Re^{v_0 n+1}$. Any suitable optimization algorithm can be applied iteratively until the penalty function J_0, (67), is reduced to zero in double precision. At each iteration of the optimization algorithm and for given $(\mathbf{p}; E_0^{(\text{nom})})$, the computation of $J_0(\mathbf{p}; E_0^{(\text{nom})})$ is done as follows. First, the control function $\mathbf{u}$ is computed via (65). Then, (38)–(52) are solved for all scenarios $s \in S$. Once $J_0(\mathbf{p}; E_0^{(\text{nom})})$ is obtained, the optimization algorithm then computes another vector $\mathbf{p}$ and number $E_0^{(\text{nom})}$ as part of its procedure.

Due to the very complicated mapping $(\mathbf{p}; E_0^{(\text{nom})}) \to J_0(\mathbf{p}; E_0^{(\text{nom})})$, (67),(65), the question of existence of solutions to equation (70) will not be dealt with here.

Since at time $k\Delta$, $k \in \{1, 2, \dots, N-1\}$, there will be updated information about predicted asset returns, the above-mentioned procedure can be applied to yield a feasible portfolio control strategy. This can be done at each time instant $k\Delta$, $k = 1, 2, \dots, N-1$.

4. COMPUTATIONAL RESULTS

The computational results are obtained using the following data. The basic time period $\Delta = 1$ month, $N = 120$, the time horizon is thus 120 months or 10 years. In addition, $r_f = (3.5/12)\%$, $\theta = 0.85$, $\lambda(k) = (0.02/12)$, $k = 1,\dots,N$, $g = (3/12)\%$, $L_0 = 1.0$, $d_0 = 0.9$.

In this case, $n = 12$ asset indices were selected from approximately 45 indices plus the risk-free asset given in [1]. The selected indices consist mainly of Italian bond market indices, foreign currency cash deposits, international bond indices, etc. (see Appendix 1 for more details).

The above data implies that there are $m = 4 + n = 16$ variables in the discrete time asset/liability model (36).

The portfolio control strategy is parametrised by using $v_0 = 5$ vectors $\mathbf{p}_i \in \Re^n \setminus \{\mathbf{0}\}$, $i = 1, \dots, v_0$. Thus, $k_0 = 24$ and the values of the reference portfolio weights can change every 24 months.

It is assumed that rebalancing takes place every month, implying $T_B = \{1, 2, \dots, N\}$, $T_Q = \emptyset$. This is done in order to prevent the portfolio weights from drifting away from the reference portfolio weights. The parameters appearing in the constraints, (57)–(62) are: $d^{(i)} = 2$, $i = \text{sh}, \text{pol}$, $e^{(\text{sh})} = 2.0144$, $e^{(\text{pol})} = 1.9284$, $\rho = 0.04$, and $\rho_1 = 0.065$. The transaction costs are computed using a commission rate $b_i = 0.005$, $i = 1, \dots, n-1$, $b_n = 0$. A total of $\nu = 100$ scenarios are used in the computations, and $p_s = 1/\nu$, $\forall\, s \in S = \{s_1, \dots, s_\nu\}$. See Appendix 2 for a description of how these scenarios were generated.

Thus, a grand total of $nv_0 + 1 = 61$ variables have to be computed, such that constraints (57)–(62) are satisfied. The penalty weights used in (67) are: $\alpha_i = 1$, $i = 1, 2, 3, 4$. Note that equation (28) is solved by using a numerical procedure described in Appendix 3.

The Nelder-Mead search based optimization algorithm (Matlab optimization toolbox) was successfully applied to compute a solution to equation (70). Thus, all the constraints and requirements have been satisfied. The computed feasible portfolio control strategy, $\mathbf{u}^*$, is plotted in Figures 2–5 and

$$E_0^{(\text{nom})*} = 0.063177. \tag{71}$$

Since portfolio rebalancing is performed at each time instant $k\Delta$, $k = 1, \dots, N$, the computed reference portfolio weights are assigned to the actual portfolio weights, that is, $\mathbf{w}(k; s) = \mathbf{u}^*(k)$, $k = 1, \dots, N$, $\forall\, s \in S$ (see (25)).

Using the definitions in (54)–(56), the following quantities are computed.

Group 1. $M(L; k)$, $M(A; k)$, and $M(L^{(\text{nos})}; k)$, $k = 0, 1, \dots, N$.
Group 2. $M(E^{(\text{nom})}; k)$, and $M(E^{(\text{res})}; k)$, $k = 0, 1, \dots, N$.
Group 3. $D(L; k)$, $D(A; k)$, and $D(L^{(\text{nos})}; k)$, $k = 0, 1, \dots, N$.
Group 4. $D(E^{(\text{nom})}; k)$ and $D(E^{(\text{res})}; k)$, $k = 0, 1, \dots, N$.

Plots of the quantities in Group 1 are shown in Figure 6, while plots of the quantities in Group 2 are shown in Figure 7. In addition, plots of the quantities in Group 3 are shown in Figure 8, and plots of the quantities in Group 4 are shown in Figure 9.

The quantities $\psi^{(\text{sh})}$, $\psi^{(\text{pol})}$ are defined here by (see (57),(58))

$$\psi^{(\text{sh})}(k) = M\left(y^{(\text{sh})}; k\right) - d^{(\text{sh})} D\left(y^{(\text{sh})}; k\right), \qquad k = 0, \dots, N, \tag{72}$$

$$\psi^{(\text{pol})}(k) = M\left(y^{(\text{pol})}; k\right) - d^{(\text{pol})} D\left(y^{(\text{pol})}; k\right), \qquad k = 0, \dots, N. \tag{73}$$

The quantities $\psi^{(\text{sh})}(k)$ and $\psi^{(\text{pol})}(k)$, $k = 0, 1, \dots, N$, are computed, (54)–(56), and plots of these quantities are shown in Figure 10. It turns out that

$$\psi^{(\text{sh})}(N) = 2.0144 \geq e^{(\text{sh})}, \qquad \psi^{(\text{pol})}(N) = 1.9285 > e^{(\text{pol})}, \qquad N = 120.$$

In addition, a variable Z is defined here by (see regulatory constraint (59))

$$Z(k; s) = \frac{A(k; s) - L(k; s)}{L(k; s)}, \qquad k = 0, 1, \dots, N, \quad \forall\, s \in S. \tag{74}$$

The quantities $M(Z; k)$ and $D(Z; k)$, $k = 0, 1, \dots, N$, are computed, (54)–(56), and plots of these quantities are shown in Figures 11 and 12, respectively. It turns out that

$$\min_{s \in S} \left(\min_{k=0,\dots,N} (Z(k; s)) \right) = 0.061276 > \rho = 0.04. \tag{75}$$

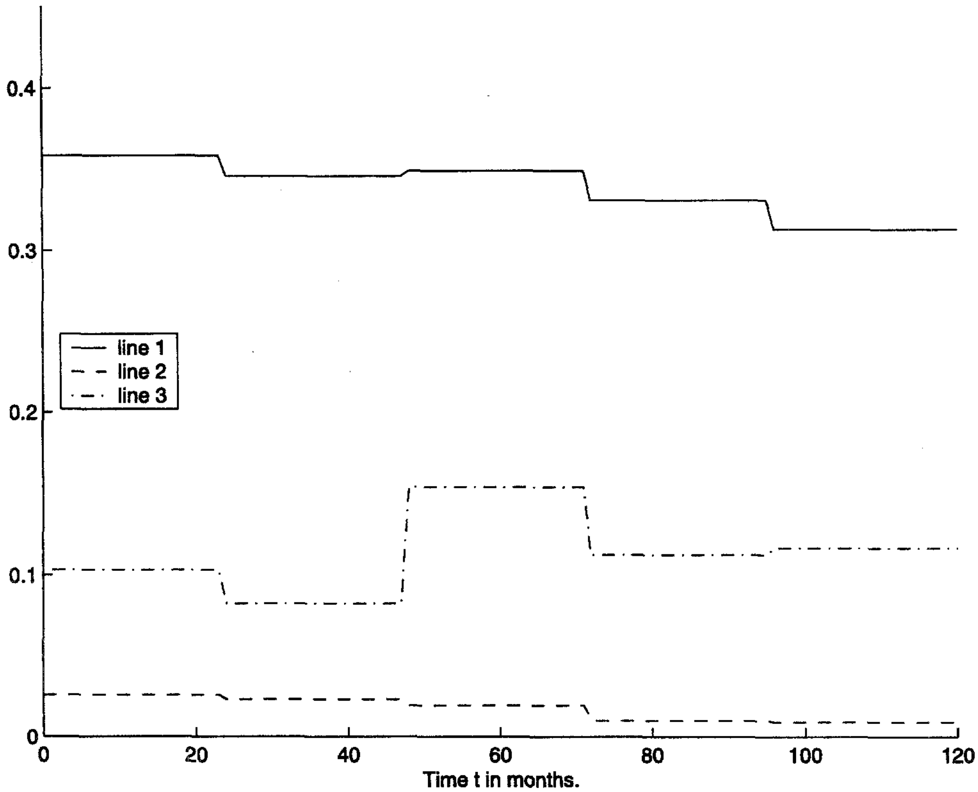

Figure 2. Plots of the reference portfolio weights $u_1^*(k)$ (line 1), $u_2^*(k)$ (line 2), $u_3^*(k)$ (line 3), versus time $t = k\Delta$, $k = 0, 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

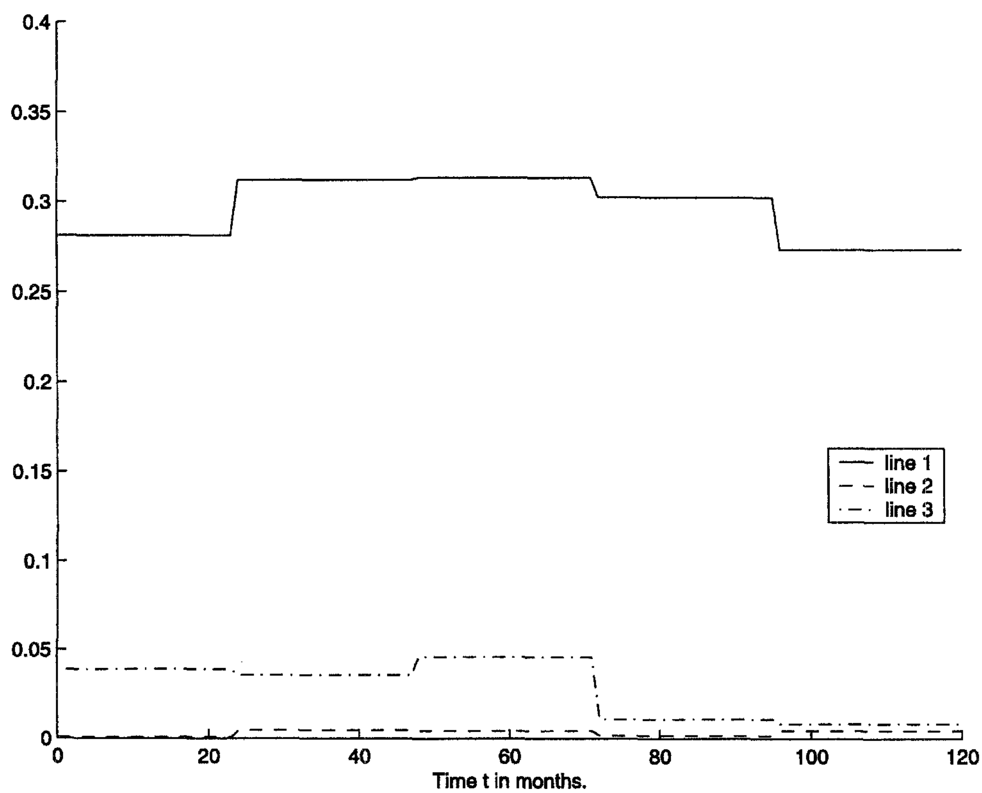

Figure 3. Plots of the reference portfolio weights $u_4^*(k)$ (line 1), $u_5^*(k)$ (line 2), $u_6^*(k)$ (line 3), versus time $t = k\Delta$, $k = 0, 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

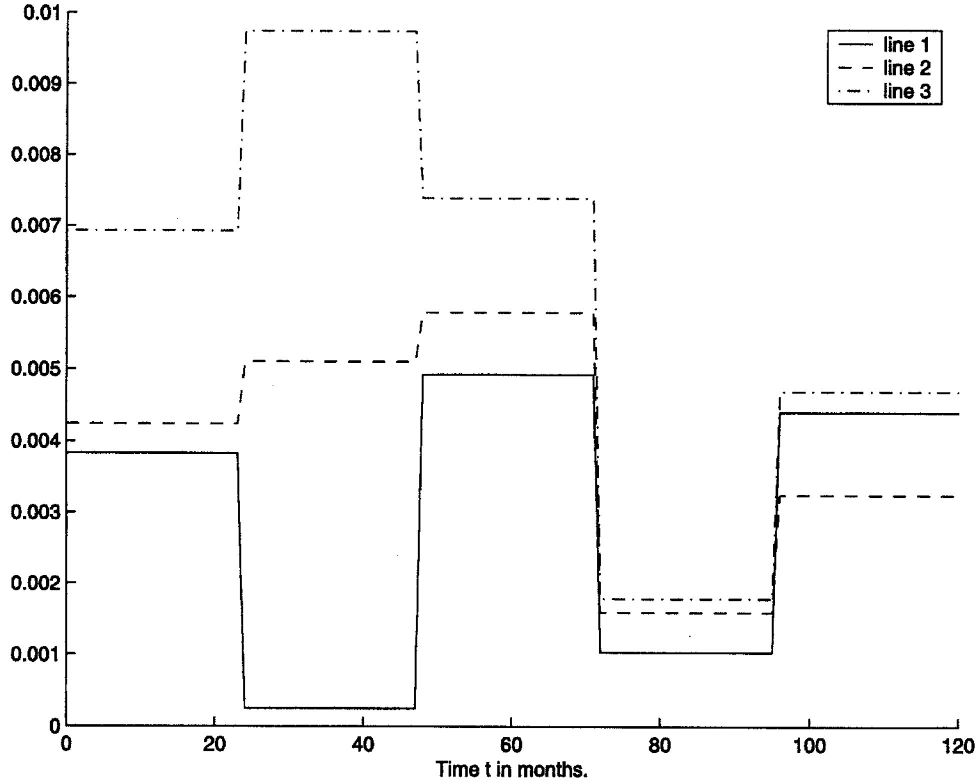

Figure 4. Plots of the reference portfolio weights $u_7^*(k)$ (line 1), $u_8^*(k)$ (line 2), $u_9^*(k)$ (line 3), versus time $t = k\Delta$, $k = 0, 1, \dots, N$, $\Delta = 1$ month, $N = 120$.

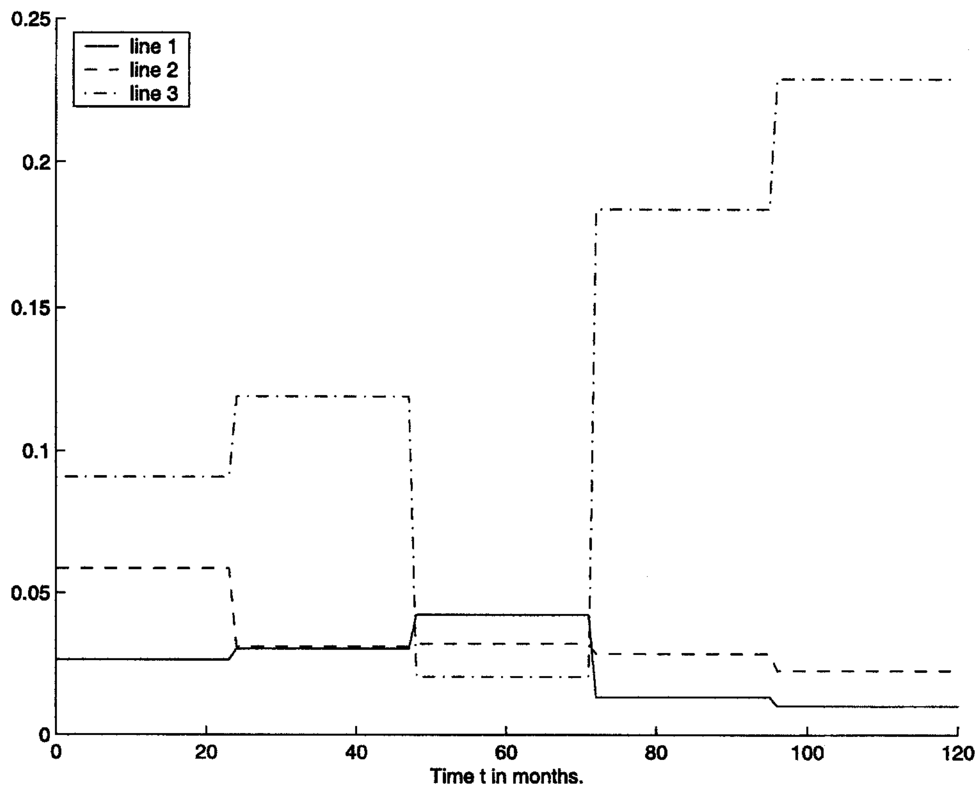

Figure 5. Plots of the reference portfolio weights $u_{10}^*(k)$ (line 1), $u_{11}^*(k)$ (line 2), $u_{12}^*(k)$ (line 3), versus time $t = k\Delta$, $k = 0, 1, \dots, N$, $\Delta = 1$ month, $N = 120$.

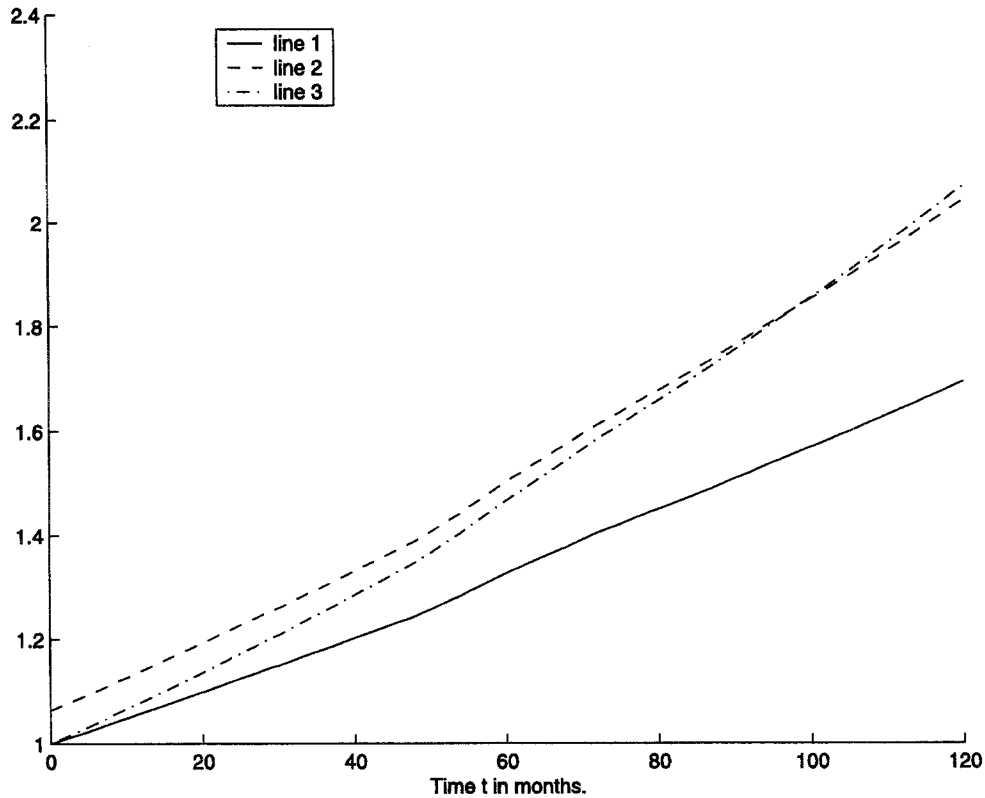

Figure 6. Plots of $M(L;k)$ (line 1), $M(A;k)$ (line 2), $M(L^{(\text{nos})};k)$ (line 3), versus time $t = k\Delta$, $k = 0, 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

The compounded portfolio rate of return up to the end of the time interval $(0, k\Delta]$, $1+R^{(c)}(k;s)$, $k = 1, 2, \ldots$, is defined here by

$$1 + R^{(c)}(k;s) = \prod_{i=1}^{k} \left(1 + R^{(p)}(i;s)\right), \qquad k = 1, \ldots, N, \quad \forall\, s \in S, \tag{76}$$

where $R^{(p)}$ is defined in (11). As above, using (54)–(56), the quantities $M(R^{(c)};k)$ and $D(R^{(c)};k)$, $k = 1, \ldots, N$, are computed, and plots of these quantities are shown in Figure 13.

The annualized portfolio rate of return at the end of the time interval $(0, k\Delta]$, $R^{(a)}(k;s)$, $k = 1, 2, \ldots$, is defined here by

$$\begin{aligned} \left[1 + R^{(a)}(k;s)\right]^{k/12} &= 1 + R^{(c)}(k;s), && k = 1, 2, \ldots, N, \quad \forall\, s \in S, \\ \Rightarrow R^{(a)}(k;s) &= \left[1 + R^{(c)}(k;s)\right]^{12/k} - 1, && k = 1, 2, \ldots, N, \quad \forall\, s \in S, \end{aligned} \tag{77}$$

provided that $1 + R^{(c)}(k;s) > 0$, $k = 1, \ldots, N$, $\forall\, s \in S$. This condition has been satisfied for all the computations performed here. The quantities $M(R^{(a)};k)$ and $D(R^{(a)};k)$, $k = 1, \ldots, N$, are computed, (54)–(56), and plots of these quantities are shown in Figures 14 and 15, respectively. It turns out that

$$M\left(R^{(a)};N\right) = 8.3918\%, \quad D\left(R^{(a)};N\right) = 0.50279\%, \qquad N = 120.$$

The annualized policyholders rate of return at the end of the time interval $(0, k\Delta]$, $Y^{(\text{pol})}(k;s)$, $k = 1, 2, \ldots$, is defined here by

$$\begin{aligned} \left[1 + Y^{(\text{pol})}(k;s)\right]^{k/12} &= y^{(\text{pol})}(k;s), && k = 1, 2, \ldots, N, \quad \forall\, s \in S, \\ \Rightarrow Y^{(\text{pol})}(k;s) &= \left[y^{(\text{pol})}(k;s)\right]^{12/k} - 1, && k = 1, 2, \ldots, N, \quad \forall\, s \in S, \end{aligned} \tag{78}$$

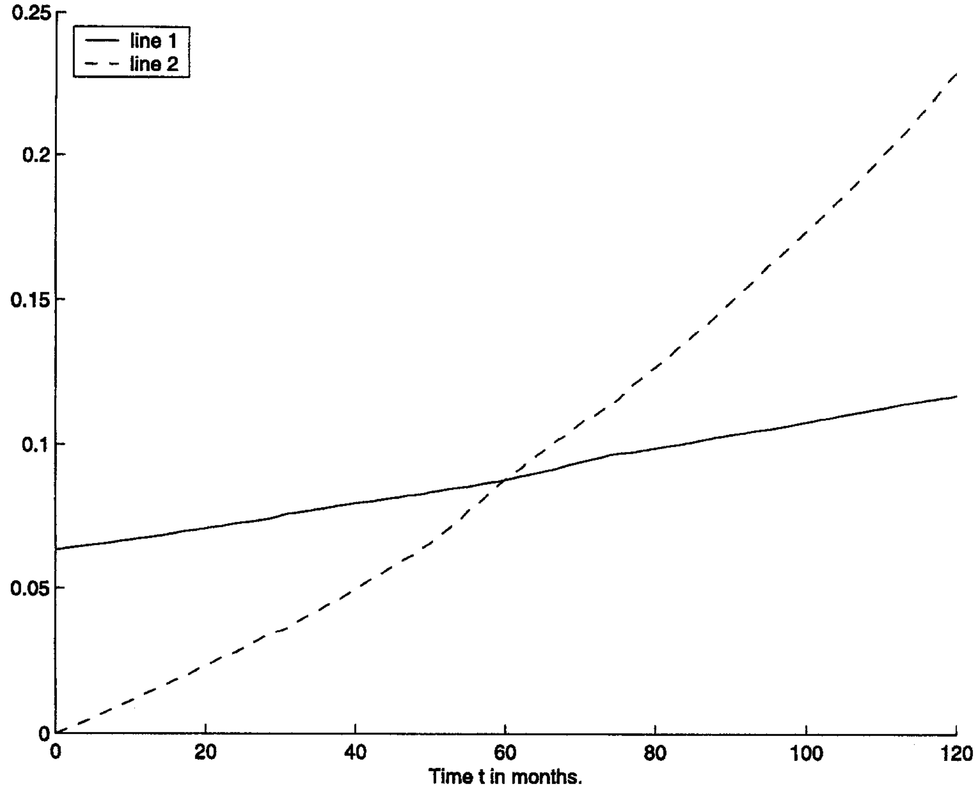

Figure 7. Plots of $M(E^{(\mathrm{nom})};k)$ (line 1), and $M(E^{(\mathrm{res})};k)$ (line 2), versus time $t = k\Delta$, $k = 0,1,\dots,N$, $\Delta = 1$ month, $N = 120$.

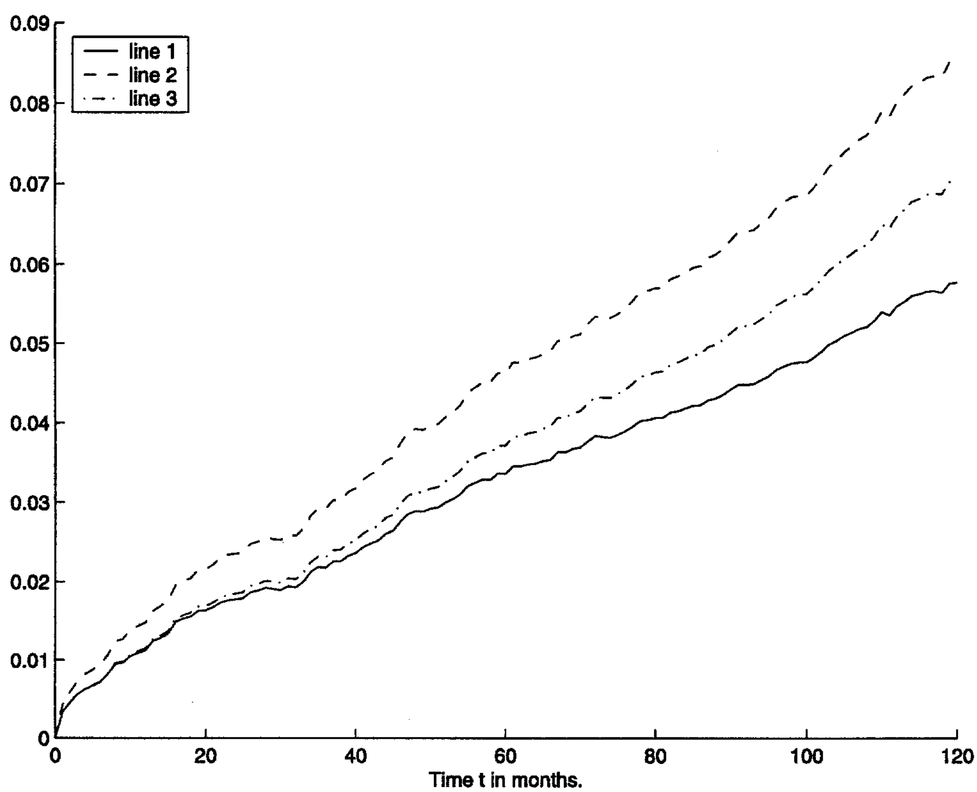

Figure 8. Plots of $D(L;k)$ (line 1), $D(A;k)$ (line 2), $D(L^{(\mathrm{nos})};k)$ (line 3), versus time $t = k\Delta$, $k = 0,1,\dots,N$, $\Delta = 1$ month, $N = 120$.

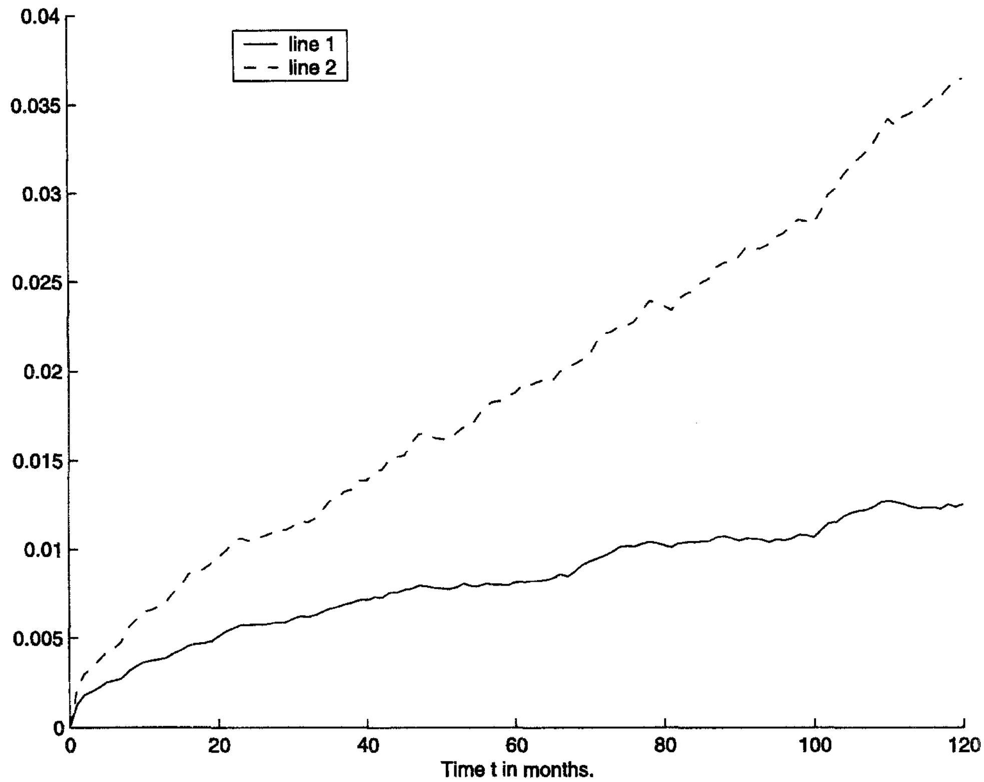

Figure 9. Plots of $D(E^{(\mathrm{nom})}; k)$ (line 1), and $D(E^{(\mathrm{res})}; k)$ (line 2), versus time $t = k\Delta$, $k = 0, 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

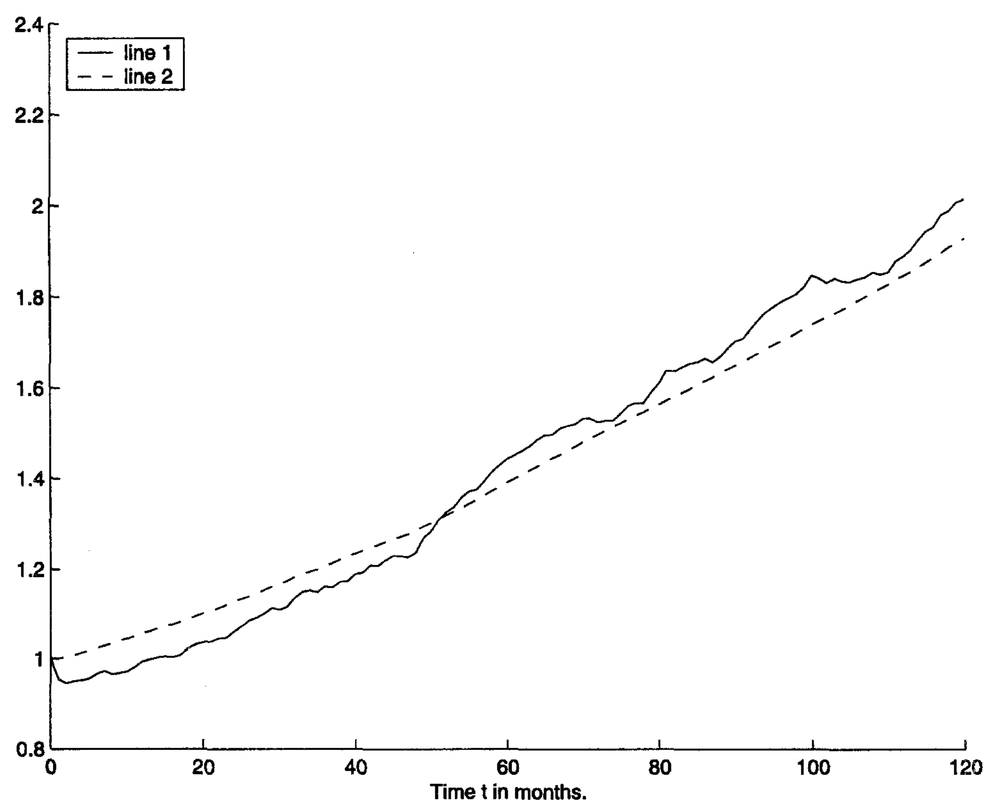

Figure 10. Plot of $\psi^{(\mathrm{sh})}(k)$ (line 1), $\psi^{(\mathrm{pol})}(k)$ (line 2), versus time $t = k\Delta$, $k = 0, 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

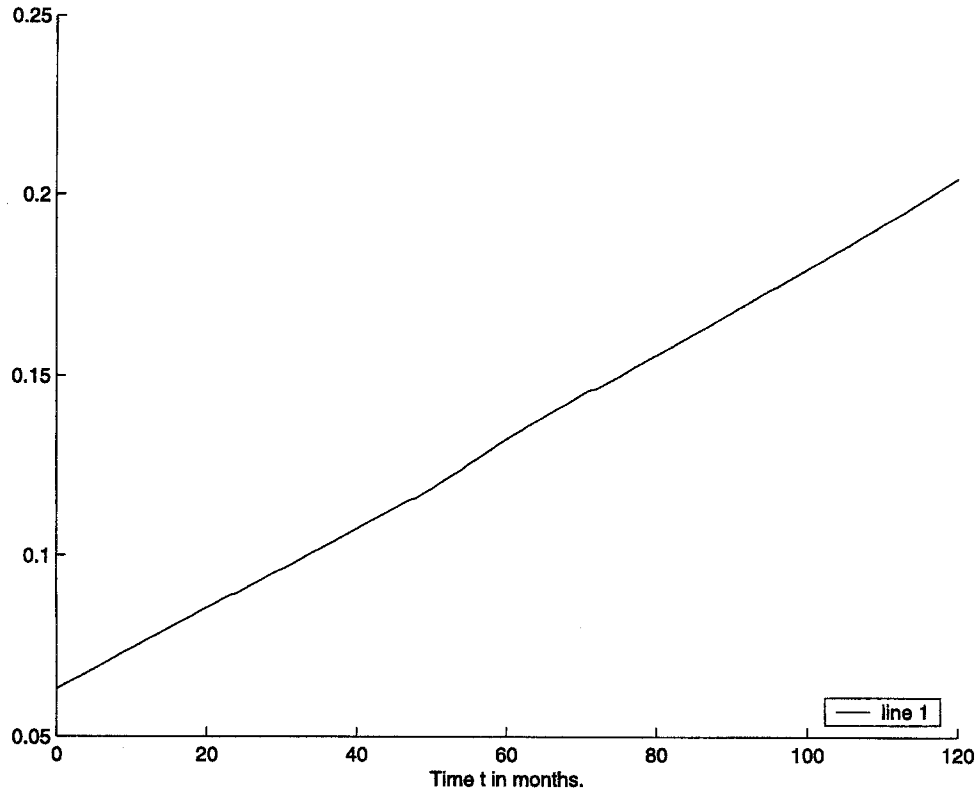

Figure 11. Plot of $M(Z;k)$ (line 1), versus time $t = k\Delta$, $k = 0, 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

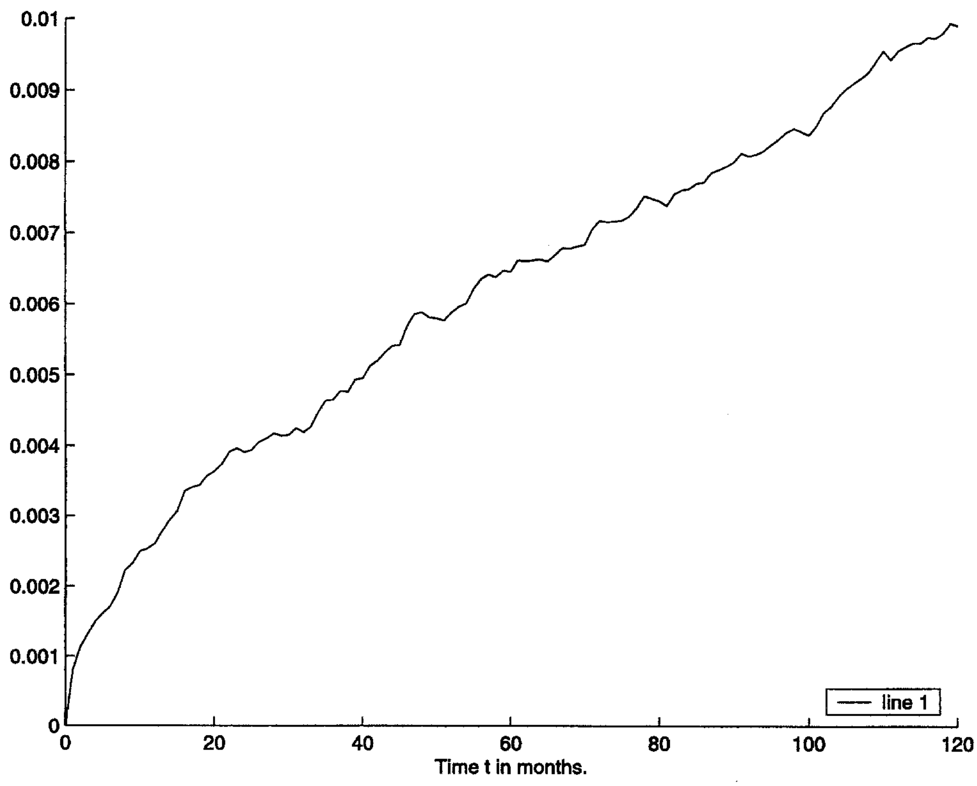

Figure 12. Plot of $D(Z;k)$ (line 1), versus time $t = k\Delta$, $k = 0, 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

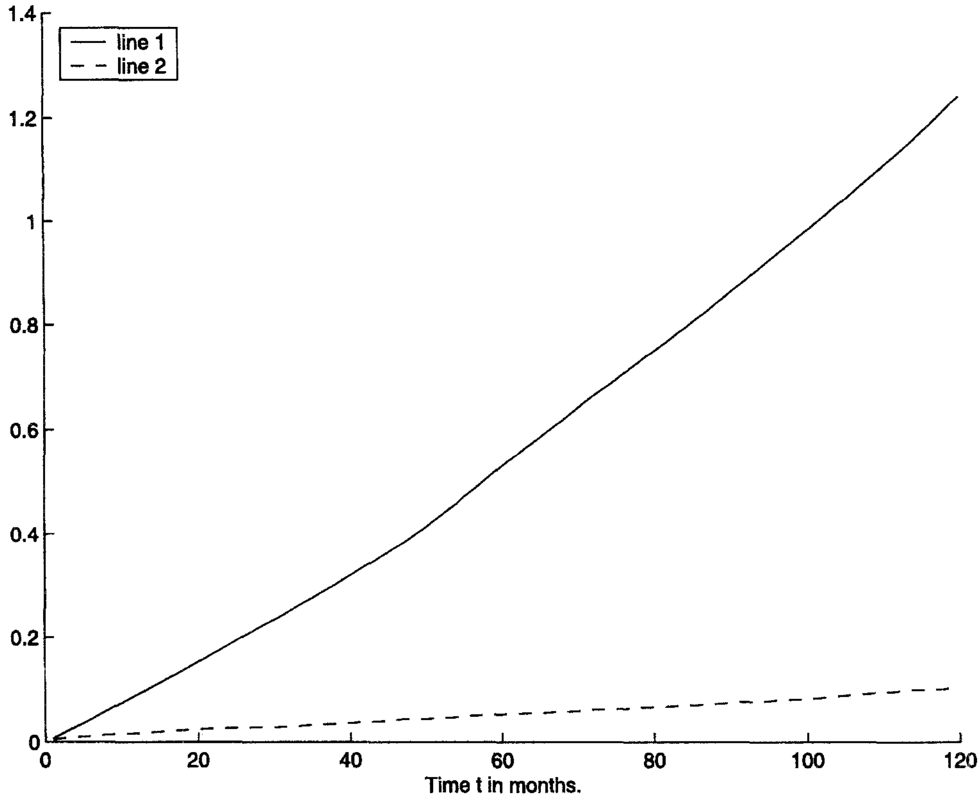

Figure 13. Plots of $M(R^{(c)};k)$ (line 1), and $D(R^{(c)};k)$ (line 2), versus time $t = k\Delta$, $k = 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

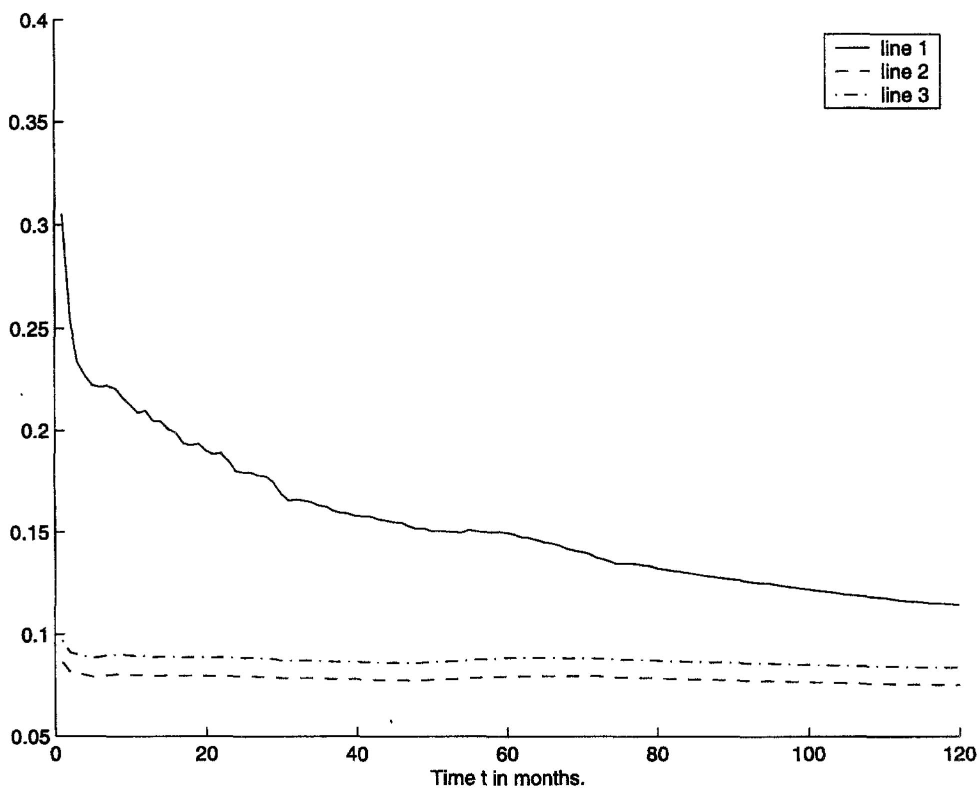

Figure 14. Plots of $M(Y^{(\mathrm{sh})};k)$ (line 1), $M(Y^{(\mathrm{pol})};k)$ (line 2), $M(R^{(a)};k)$ (line 3), versus time $t = k\Delta$, $k = 1, \ldots, N$, $\Delta = 1$ month, $N = 120$.

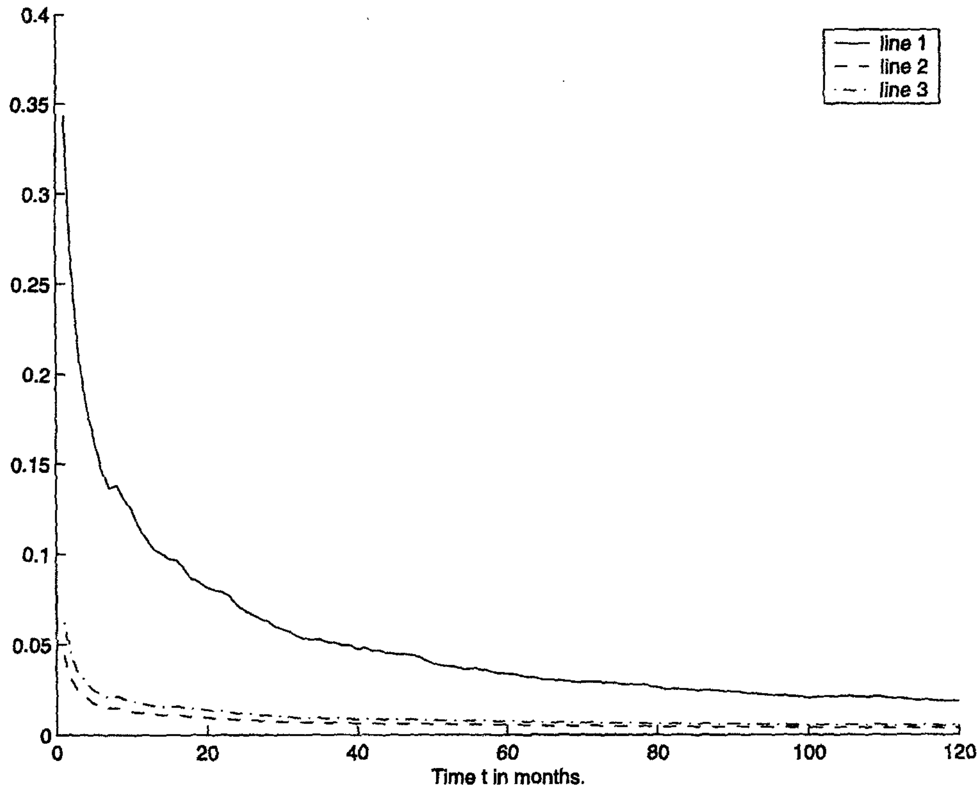

Figure 15. Plots of $D(Y^{(\mathrm{sh})};k)$ (line 1), $D(Y^{(\mathrm{pol})};k)$ (line 2), $D(R^{(a)};k)$ (line 3), versus time $t = k\Delta$, $k = 1,\dots,N$, $\Delta = 1$ month, $N = 120$.

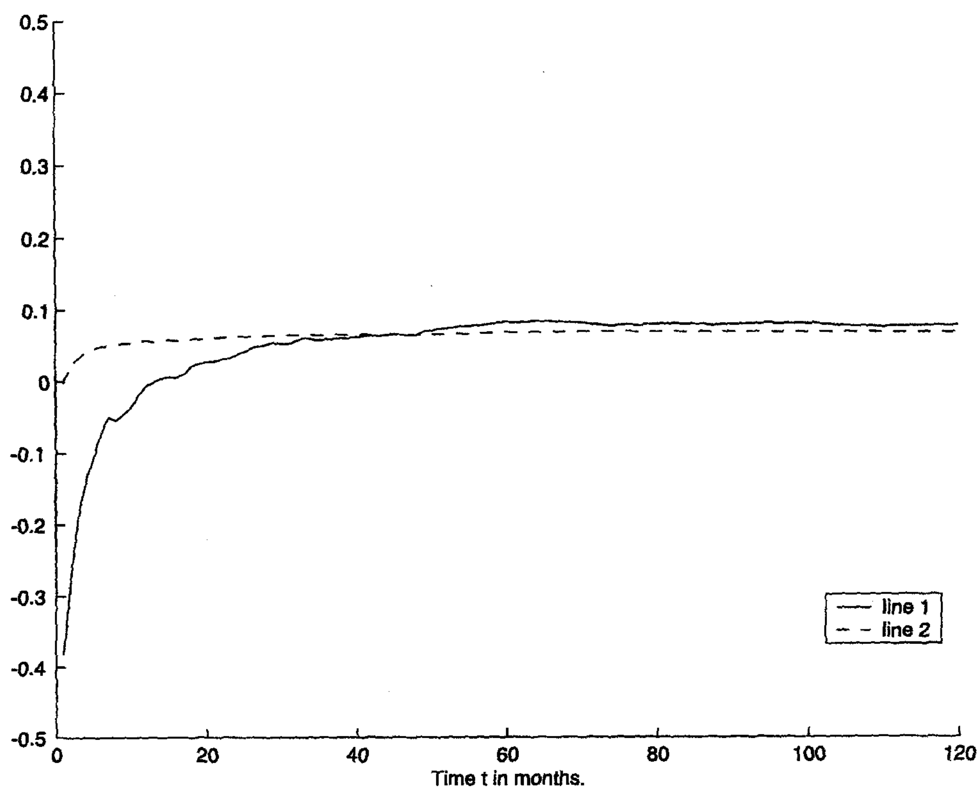

Figure 16. Plots of $\Gamma^{(\mathrm{sh})}(k)$ (line 1), $\Gamma^{(\mathrm{pol})}(k)$ (line 2), versus time $t = k\Delta$, $k = 1,\dots,N$, $\Delta = 1$ month, $N = 120$.

provided that $y^{(\mathrm{pol})}(k;s) > 0$, $k = 1, \ldots, N$, $\forall\, s \in S$. Similarly, the annualized shareholders rate of return at the end of the time interval $(0, k\Delta]$, $Y^{(\mathrm{sh})}(k;s)$, $k = 1, 2, \ldots$, is defined here by

$$Y^{(\mathrm{sh})}(k;s) = \left[y^{(\mathrm{sh})}(k;s)\right]^{12/k} - 1, \qquad k = 1, 2, \ldots, N, \quad \forall\, s \in S, \tag{79}$$

provided that $y^{(\mathrm{sh})}(k;s) > 0$, $k = 1, \ldots, N$, $\forall\, s \in S$. The conditions $y^{(i)}(k;s) > 0$, $k = 1, \ldots, N$, $\forall\, s \in S$, $i = \mathrm{sh}, \mathrm{pol}$, have been satisfied for all the computations performed here.

The quantities $M(Y^{(\mathrm{pol})};k)$, $M(Y^{(\mathrm{sh})};k)$, $k = 1, \ldots, N$, and $D(Y^{(\mathrm{pol})};k)$, $D(Y^{(\mathrm{sh})};k)$, $k = 1, \ldots, N$, are computed, (54)–(56), while plots of these quantities are shown in Figures 14 and 15, respectively.

It turns out that

$$M\left(Y^{(\mathrm{pol})};N\right) = 7.5395\%, \quad D\left(Y^{(\mathrm{pol})};N\right) = 0.36828\%, \qquad N = 120,$$
$$M\left(Y^{(\mathrm{sh})};N\right) = 11.484\%, \quad D\left(Y^{(\mathrm{sh})};N\right) = 1.8423\%, \qquad N = 120.$$

In addition, $\Gamma^{(\mathrm{pol})}(k)$, $\Gamma^{(\mathrm{sh})}(k)$ are defined here by

$$\Gamma^{(\mathrm{pol})}(k) = M\left(Y^{(\mathrm{pol})};k\right) - d^{(\mathrm{pol})} D\left(Y^{(\mathrm{pol})};k\right), \qquad k = 1, \ldots, N, \tag{80}$$

$$\Gamma^{(\mathrm{sh})}(k) = M\left(Y^{(\mathrm{sh})};k\right) - d^{(\mathrm{sh})} D\left(Y^{(\mathrm{sh})};k\right), \qquad k = 1, \ldots, N, \tag{81}$$

(see (78),(79)), and plots of these quantities are shown in Figure 16.

The software for performing all the above computations is written in MATLAB.

5. CONCLUSION

A nonlinear discrete time asset/liability model is developed for an insurance company selling investment policies with a guaranteed minimum rate of return and a fixed maturity date.

The asset/liability model accommodates time-dependent investment strategies and transaction costs. At time instants where portfolio rebalancing takes place, the model implements a constraint equation dictating that the total value of assets sold must be equal to the total value of assets purchased plus the total transaction costs. Asset transactions are thus self-financing and no additional cash is required. At time instants where no portfolio rebalancing takes place, the actual portfolio weights drift according to the predicted trajectory of asset returns.

Feasible control is applied to compute a time-dependent portfolio control strategy and the initial shareholders capital, such that the insurance company satisfies given financial constraints and requirements.

The proposed asset/liability model is flexible and can be adapted to model insurance companies with more complicated balance sheet structures, different types of investment policies, etc. Additional financial constraints and requirements can be added to the formulation of the feasible portfolio control problem. For example, constraints incorporating the operational objectives of the insurance company, regulatory requirements, constraints representing the interests of policyholders and shareholders, and other requirements.

APPENDIX 1

A total of 45 asset indices representing various asset sectors, mainly in Italy, and also internationally, are given in [1]. The risk-free asset is taken to be a bank cash deposit account in Italian Lire. Twelve asset indices are selected (listed below) using the following procedure.

1. Italian Government Bond Index 1–3 years (YRS-1-3).

2. Italian Government Bond Index 3–7 years (YRS-3-7).
3. Euro cash deposit (CASH-EU).
4. US Dollar cash deposit (CASH-US).
5. German Government Bond Index (GVT-GM).
6. Italian Government Bond Index (GVT-IT).
7. French Government Bond Index (GVT-FR).
8. Spanish Government Bond Index (GVT-SP).
9. US Government Bond Index (GVT-US).
10. UK Government Bond Index (GVT-UK).
11. Sectional Corporate Bond Index: Life Insurance Industry (CRP-LFE).
12. Risk Free Asset taken as an Italian Lire cash deposit account (IT-RiskFree).

FIRST. The feasible portfolio control problem is solved for the case where all 45 indices plus the risk-free asset are used. In this case, the portfolio control strategy $\mathbf{u}$ is taken as constant over the time horizon $[0, N\Delta]$, that is, $v_0 = 1$, $k_0 = N = 120$ (see (65)). In this computation, use is made of the parameters given in Section 4, except for the parameters $e^{(\text{sh})} = e^{(\text{pol})} = 1.827$, (57),(58), and $\rho_1 = 0.2$, (60).

It turns out that the reference portfolio weights of about seven asset indices are greater than or equal to 1%, while the rest are negligible with respect to 1%. The 12 indices listed above include most of the afore-mentioned seven indices with significant portfolio weights.

APPENDIX 2

Monthly index values of a total of $n_T = 45$ asset indices are available from the beginning of January 1990 to the beginning of January 2000 [1].

Assume that n_0, $n_0 \leq n_T$, of the available asset indices are selected. Number the selected asset indices $1, 2, \dots, n_0$, and denote the value of index i at time instant $j\Delta_0$ by $I_i(j)$, $i = 1, \dots, n_0$, $j = 0, \dots, N_0 = 120$, where $\Delta_0 = 1$ month, $j = 0$ corresponds to the beginning of January 1990, and $j = N_0$ corresponds to the beginning of January 2000.

The rate of return of asset index i at the end of the one month time interval $(j, j+1]$, $h_i(j+1)$, is defined here by

$$h_i(j+1) = \frac{I_i(j+1) - I_i(j)}{I_i(j)}, \qquad i = 1, \dots, n_0, \quad j = 0, 1, \dots, N_0 - 1. \tag{82}$$

Define the vectors $\mathbf{h}(j)$, $j = 1, \dots, N_0$,

$$\mathbf{h}(j) = [h_1(j), \dots, h_{n_0}(j)]^\top, \qquad j = 1, \dots, N_0. \tag{83}$$

It is assumed that $\mathbf{h}(j)$, $j = 1, \dots, N_0$, are measurements of a sequence of independent, identically distributed random vectors, each having an n_0-variate Gaussian density, with mean vector, $\boldsymbol{\mu}_A$, and covariance matrix, $\boldsymbol{\Sigma}_A$. Using this assumption, the measurements $\mathbf{h}(j)$, $j = 1, \dots, N_0$, are employed to compute estimates for the mean vector, $\boldsymbol{\mu} = [\mu_1, \dots, \mu_{n_0}]^\top$, and covariance matrix, $\boldsymbol{\Sigma} = [\Sigma_{ik}]$, as follows:

$$\mu_i = \frac{1}{N_0} \sum_{j=1}^{N_0} h_i(j), \qquad i = 1, \dots, n_0, \tag{84}$$

$$\Sigma_{ik} = \frac{1}{N_0} \sum_{j=1}^{N_0} h_i(j) h_k(j) - \mu_i \mu_k, \qquad i, k = 1, \dots, n_0. \tag{85}$$

In this case, the first 11 asset indices listed in Appendix 1 are selected ($n_0 = 11$).

For a given scenario $s \in S$, the predicted trajectory of asset returns, $\mathbf{r}(k; s) \in \Re^n$, $k = 1, \dots, N = 120$, $n = n_0 + 1 = 12$, is generated by using the following procedure.

Consider $k = 1$, and employ a pseudorandom number generator to compute a sample of an $(n-1)$-dimensional random vector having a Gaussian density with mean vector, $\boldsymbol{\mu}$, and covariance matrix, $\boldsymbol{\Sigma}$. This sample is assigned to the first $(n-1)$ components of $\mathbf{r}(1;s)$, while the risk-free rate is assigned to the last component, (2). The afore-mentioned procedure is then repeated to compute $\mathbf{r}(k;s)$, $k = 2,\dots,N$.

Note that if for any $k \in \{1,\dots,N\}$, and any $i \in \{1,\dots,n-1\}$, $r_i(k;s) \le -1$, then additional samples are generated until the afore-mentioned condition does not occur.

Thus, the total procedure described above is repeated in order to generate all the scenarios, $\mathbf{r}(k;s) \in \Re^n$, $k = 1,\dots,N$, $s = s_1,\dots,s_\nu$.

Other methods for generating scenarios are given in [1] and the references cited there.

APPENDIX 3

For $k \in \{0,1,\dots,N-1\}$ and $(k+1) \in T_B$, and for a given $s \in S$, the following equation, (28), has to be solved for $A(k+1;s)$,

$$A(k+1;s) = A\left((k+1)^-;s\right) - \sum_{i=1}^{n} b_i \left|u_i(k+1)A(k+1;s) - X_i\left((k+1)^-;s\right)\right|, \tag{86}$$

subject to the constraint, (29),

$$A(k+1;s) > 0. \tag{87}$$

The solution of (86) is obtained by using the following iterative algorithm:

$$\begin{gathered} h^{(j+1)} = A\left((k+1)^-;s\right) - \sum_{i=1}^{n} b_i \left|u_i(k+1)h^{(j)} - X_i\left((k+1)^-;s\right)\right|, \\ j = 0,1,2,\dots,\min[j_{\max}, j_c], \end{gathered} \tag{88}$$

where

$$h^{(0)} = A\left((k+1)^-;s\right), \tag{89}$$

$j_{\max} > 0$ is a given integer, and j_c is the iteration number where the following is satisfied:

$$\left|h^{(j_c+1)} - h^{(j_c)}\right| \le \epsilon, \qquad 0 < \epsilon \ll 1. \tag{90}$$

If the convergence criterion (90) is satisfied for $j_c \le j_{\max}$ and

$$h^{(j_c+1)} > 0, \tag{91}$$

then $h^{(j_c+1)}$ is taken to be the solution of (86), that is,

$$A(k+1;s) = h^{(j_c+1)}. \tag{92}$$

The following parameter values are used: $j_{\max} = 1000$, $\epsilon = 10^{-20}$.

It turns out that the convergence criterion (90) is satisfied for $j_c \ll j_{\max}$, and (91) is met for all the computations performed in Section 4.

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