

Pricing Options on Scenario Trees*

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Abstract

We examine valuation procedures that can be applied to incorporate options in scenario-based portfolio optimization models. Stochastic programming models use discrete scenarios to represent the stochastic evolution of asset prices. At issue is the adoption of suitable procedures to price options on the basis of the postulated discrete distributions of asset prices so as to ensure internally consistent portfolio optimization models. We adapt and implement two methods to price European options in accordance with discrete distributions represented by scenario trees and assess their performance with numerical tests. We consider features of option prices that are observed in practice. We find that asymmetries and/or leptokurtic features in the distribution of the underlying materially affect option prices; we quantify the impact of higher moments (skewness and excess kurtosis) on option prices. We demonstrate through empirical tests using market prices of the S&P500 stock index and options on the index that the proposed procedures consistently approximate the observed prices of options under different market regimes, especially for deep out-of-the-money options.

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1 Introduction

Stochastic programs provide an effective framework for modelling diverse financial management problems. The deployment of stochastic programming models in practice is made possible due to flexible modelling systems, algorithmic developments for large-scale stochastic programs and advancements of computing technology. As a result, stochastic programs are increasingly gaining acceptance as viable tools for addressing diverse financial planning problems under uncertainty. Recent advancements of models and applications of stochastic programs for portfolio management and related problems are documented, for example, in [42, 15, 41, 38, 39, 36, 37].

Stochastic programs are gaining popularity because of their flexibility and several advantageous features. Multistage stochastic programs provide an effective basis to model dynamic portfolio management problems. They can incorporate practical considerations such as portfolios of many assets,

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transaction costs, liquidity, trading and turnover constraints, limits on holdings in individual assets or groups, as well as additional managerial and regulatory requirements that can be modelled as constraints on the decision variables. Moreover, stochastic programs can accommodate different objective functions to represent the decision maker’s risk bearing attitude and the primary decision goals. These features provide substantial leeway in modelling practical financial planning problems.

In dynamic stochastic programs uncertainty in input parameters is flexibly modelled in terms of discrete distributions. The discrete distributions can capture the joint co-variation of the random variables and are represented by means of a scenario tree that depicts the progressive evolution of the random variables. The models are not restricted by rigid distributional assumptions. Asymmetric and heavy-tailed distributions of random variables, that are often implied by financial times series, can be accommodated in scenario-based models.

These flexibilities of stochastic programs make them suitable tools for risk management applications. It is natural to consider, in the context of stochastic programs, appropriate instruments that can serve the risk management function. Derivatives are well-suited for this purpose. By construction, options have asymmetric payoffs that can cover against adverse price movements of the underlying security. Combinations of options can help shape a desired payoff profile over the range of possible movements in the price of the underlyings. Consequently, the incorporation of options in portfolio optimization models should lead to improved risk management tools.

It is not our intention to provide a comprehensive review of the extensive literature on option pricing; we only cite some representative works.

In their seminal work, Black and Scholes [5] derived explicit pricing formulas for both call and put options based on the assumption that stock prices follow a geometric Brownian motion. Several empirical studies (e.g., Rubinstein [31, 32]) showed that the B-S model misprices deep out-of-the money options. The volatility estimates in the Black-Scholes formula, implied by market prices of options and their underlying securities, differ across exercise prices and maturities and form “smile” patterns, violating the constant volatility assumption. Empirical estimates of the risk-neutral probability density of asset returns reveal (often negatively) skewed and leptokurtic distributions, in contrast to the log-normal distribution assumed in the Black-Scholes model.

Dempster and his collaborators [9, 10, 11] developed, implemented and tested procedures based on linear programming for pricing American and exotic options in the Black-Scholes framework. Their procedures exploit the problem structure in specializing operations of the revised simplex algorithm to solve efficiently finite difference approximations of the Black-Scholes differential equation.

Another option pricing approach infers the risk-neutral distribution for the price of the underlying security. The observed financial time series and risk premia imply a risk-neutral probability measure that can be used to price any derivative as the expected discounted value of its future payoff. While the physical (empirical, subjective) and risk-neutral probability measures are related, they are identical only in the case of zero premia on all relevant risks factors. A fundamental theorem of asset pricing states that in the absence of arbitrage there exists some pricing kernel that can reconcile the two measures. This result prompted the studies by Aït-Sahalia and Lo [1], Jackwerth [19], and Rosenberg and Engle [30] regarding the characteristics of the pricing kernel.

An alternative approach to deal with nonconstant volatility was proposed by Rubinstein [32], Jackwerth and Rubinstein [20], and in a series of papers by Derman and Kani [12], Derman et al. [13] and Dupire [16]. Instead of imposing a parametric function for volatility, they approximate the structure of asset prices with binomial or trinomial lattices which are calibrated on the basis of market prices. Rubinstein [32] showed how to compute the implied distribution using quadratic programming; Jackwerth and Rubinstein [20] generalized this approach using nonlinear programming to minimize four different objective functions.

To incorporate options in scenario-based portfolio optimization models the options must be priced consistently with the price scenarios for the underlying securities. The following key issue needs to be resolved. Common option pricing methods typically rely on specific assumptions regarding the

stochastic process of the underlying security’s price. On the other hand, stochastic programming models afford the flexibility to cope with general discrete distributions. It is often desirable, and even necessary, to capture in the scenario sets skewness and excess kurtosis features that are often observed in market data. Hence, the postulated scenarios for the asset prices do not necessarily conform to distributional assumptions on which popular option valuation methods are based. As a result, these option pricing methods cannot be employed if the options are to be incorporated in stochastic programming models that use a different representation of uncertainty for the prices of the underlying assets. Otherwise, the models can give rise to arbitrage opportunities and imply spurious profits. Appropriate option pricing procedures must be adopted to ensure internal consistency of the stochastic programming model.

This paper aims to adapt, implement and empirically validate appropriate methods to price options in accordance with discrete scenario sets of asset prices. We confine our attention to European options and propose two procedures to price the options consistently with the discrete distributions of a scenario tree. The adopted pricing procedures account for the statistical characteristics of asset returns (including skewness and kurtosis) as reflected in their empirical distributions. A key concern is to ensure that the scenarios of asset prices, in conjunction with the resulting option prices, satisfy the fundamental no-arbitrage condition.

In the first approach, we determine an equivalent risk-neutral probability measure from the physical probability measure that is associated with the postulated price outcomes on the scenario tree. For European options, the critical input for valuation is the distribution of the underlying asset’s price (and, consequently, the option payoff) at the time of the option’s maturity. In our construction, the discrete support of this distribution is represented by a corresponding subset of nodes of the scenario tree. Once the risk-neutral probabilities of these nodes (states) are determined, pricing the options is straightforward. The price of a European option is the expectation of its discounted payoff. The expectation is taken, under the risk neutral probability measure, over the nodes of the scenario tree that represent the discrete support of asset prices at the option’s maturity.

The second approach is based on a series expansion of the distribution of asset log-returns. The series expansion allows to analytically compute the option price as a sum of three parts: the Black-Scholes price and two additional adjustment terms that account for skewness and kurtosis, as those are reflected in the scenario sets of asset prices.

The proposed procedures can price European options at any node of the scenario tree, thus allowing transactions with options at any decision stage of a multi-period portfolio optimization model. These approaches are not restricted by explicit assumptions for the distribution of the underlying security’s price; in principle, they can be applied to any arbitrary discrete distribution. Of course the appropriateness (i.e., satisfaction of no-arbitrage conditions) and the effectiveness of the methods to properly price options needs to be verified in each case.

We describe implementations of the proposed pricing procedures and assess their viability and performance through extensive numerical tests using market data. In the context of these tests, we generate scenarios of asset prices using a moment matching method. The pricing procedures, however, are not restricted to this specific scenario generation method. We verify numerically that the scenarios of asset prices satisfy the fundamental no-arbitrage condition.

We compare prices computed with the proposed procedures against observed market prices of European options on the *S&P500* stock index. We also compute estimates of the option prices with the Black-Scholes method as a reference point in comparisons. The proposed procedures produce option prices that are closer to observed market prices than the corresponding B-S prices, especially for deep out-of-the-money (OTM) options. The results show that the returns of the *S&P500* index during the time window of the numerical tests cannot be closely approximated by a log-normal distribution as is assumed in the B-S framework. This assumption is often violated by market prices and, as expected, the B-S estimates of option prices are not accurate under these circumstances.

Out-of-the-money put options are particularly suitable for risk management purposes as they

insure against severe adverse movements in the price of the underlying asset. Our results confirm that the Black-Scholes formulas systematically (and substantially) overprice deep OTM call options and underprice deep OTM puts. This observation was a primary motivation for our study that seeks to adapt appropriate methods that can reliably price options in accordance with general discrete distributions of asset prices, so as to incorporate options in stochastic optimization models geared towards risk management applications.

We demonstrate through empirical tests that the pricing of European options based on scenario trees, that capture asymmetries and heavy tails of the empirical distribution of asset prices, is a viable valuation alternative. When no-arbitrage conditions are met by the price scenarios the computed option prices are quite close to observed market prices.

The paper is organized as follows. In section 2 we discuss the structure of the scenario tree that models uncertain asset prices; we refer to the scenario generation procedure and numerical tests to verify that the price scenarios are arbitrage-free. In section 3 we present two procedures for pricing options based on the scenario tree. In section 4 we investigate the effects of higher moments of the underlying asset's distribution on the resulting option prices. We also examine features of option prices that are typically observed in practice and give rise to the volatility smile; these can be explained, at least partly, by negatively skewed risk-neutral distributions that can be captured by the proposed approaches. In section 5 we compare option prices on the *S&P500* index computed with the proposed procedures to observed market prices. Conclusions are summarized in section 6.

2 Representation of Stochastic Asset Prices on a Scenario Tree

In stochastic programming models for portfolio management the dynamic evolution of asset prices is represented in terms of a scenario tree, of the form shown in Figure 1. The root node corresponds to the current state (at time $t = 0$) from which a number of branches emanate. Each branch leads to an immediate successor node that represents an outcome that the prices of all assets jointly take at the next decision time, with an associated probability. The same principle applies to each subsequent state, corresponding to a distinct node of the scenario tree. For each node, the collection of immediate successor nodes models the conditional discrete distribution of asset prices in the next time period, contingent on their common predecessor node. Decisions (i.e., potential portfolio revisions) are considered at the nodes (states) of each time period $t = 0, 1, \dots, T$. Each node of the tree corresponds to a joint plausible outcome of the asset prices at the respective time period; thus, correlations between asset prices can be captured in the scenario generation process and reflected on the scenario tree.

We define the following notation:

\mathbf{N}	the set of nodes of the scenario tree.
$n \in \mathbf{N}$	a typical node of the scenario tree ($n = 0$ denotes the root node at $t = 0$).
$\mathbf{N}_T \subset \mathbf{N}$	the set of leaf (terminal) nodes at the last period T that uniquely identify the complete set of scenarios over the planning horizon. For each time period, a scenario corresponds to a path from the root to a distinct node at the respective period.
$p(n) \in \mathbf{N}$	the unique predecessor node of node $n \in \mathbf{N}$.
$\mathbf{M}_n \subset \mathbf{N}$	the set of immediate successor nodes of node $n \in \mathbf{N} \setminus \mathbf{N}_T$. This set of nodes models the discrete outcomes of the random asset prices at the respective time period, conditional on the predecessor node n .
S_i^n	the price of asset i , $i = 1, \dots, I$, at node $n \in \mathbf{N}$.
π_n	the unconditional probability for the outcome (state) associated with node $n \in \mathbf{N}$, as resulting from the scenario generation process. Obviously, $\sum_{n \in \mathbf{N}_T} \pi_n = 1$ and, recursively, $\pi_n = \sum_{m \in \mathbf{M}_n} \pi_m$, $\forall n \in \mathbf{N} \setminus \mathbf{N}_T$.

The scenario tree typically includes many nodes, and it is not binomial as shown in Figure 1

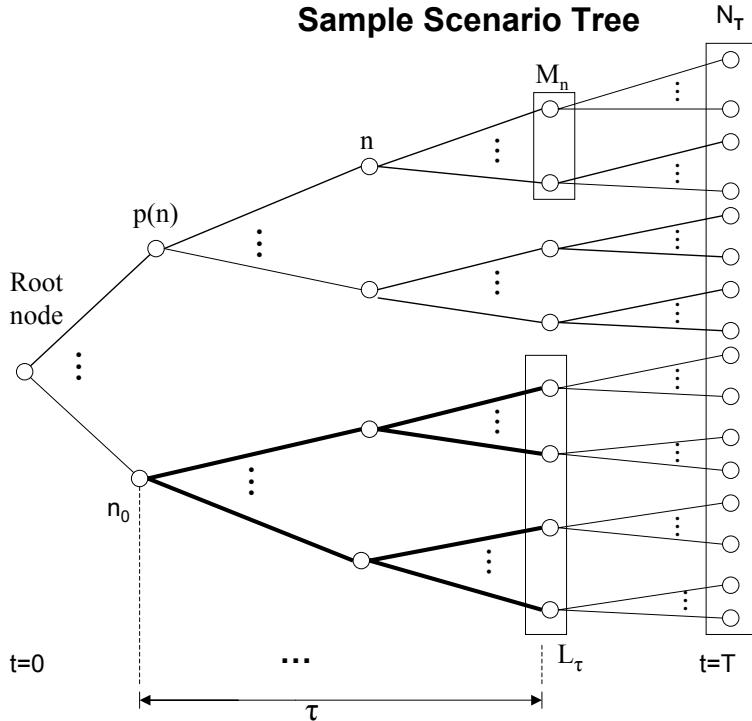


Figure 1: Illustration of a scenario tree.

simply for illustration purposes. The tree reflects, in terms of conditional discrete distributions, the dynamics of a stochastic process for the asset prices. Fine representations of the stochastic price process imply an increasing number of outcomes (branching factor from nodes) at each stage. This has a direct impact on the size of the resulting stochastic program and the required solution effort. A balance is needed between a fine representation of the stochastic price process and the computational complexity of the associated stochastic program.

The topic of scenario generation for stochastic programs is a subject of active research. Various approaches have been proposed, including judicious sampling or discretization of continuous stochastic processes, simulation and clustering, econometric models, and even the incorporation of subjective (expert) forecasts. Dupačová et al. [14] review scenario generation methods. There is no universal choice for a scenario generation method. Clearly, the set of scenarios must adequately approximate the stochastic process governing the underlying random variables. Thus, the suitability of a scenario generation method for a particular application should always be checked by the modeler before adopting any particular approach. Kaut and Wallace [25] discuss procedures to numerically evaluate scenario sets for stochastic programs.

We apply a moment-matching scenario generation method developed by Høyland et al. [18]. The method generates scenarios so that key statistics of the random variables match specified target values. We match the following statistics: the first four marginal moments (mean, variance, skewness, and kurtosis) of asset returns and, additionally, their correlations — in the case of multi-asset problems. We estimate the target values to be matched on the basis of historical market data.

A necessary condition for the asset price scenarios is the no-arbitrage requirement. The presence of arbitrage in the outcomes represented by the scenario tree would lead to solutions of the portfolio optimization program that imply spurious profits. Such solutions are of no practical usefulness. The absence of arbitrage is also a fundamental condition for option pricing. Klaasen [27] describes the conditions that asset price scenarios must satisfy to be consistent with financial asset pricing theory (i.e., to be arbitrage-free and consistent with observed market prices).

We applied the tests in Klaassen [28] to numerically verify that the asset price scenarios used in the empirical tests are arbitrage-free. The tests require the solution of two linear programs using as inputs the asset price outcomes of the scenario tree and having as decision variables the asset allocations at each node of the scenario tree. An analogous stochastic linear program to test for the absence of arbitrage in asset prices on a scenario tree is explained in Zhao and Ziemba [40].

The purpose of the tests is to ensure that the asset price scenarios do not include any “free lunch.” That is, to verify that there is no feasible costless, self-financing policy (including short sales) that can generate a riskless profit (i.e., a non-positive initial endowment that can result in non-negative payoffs at all terminal nodes of the scenario tree, and a strictly positive value at, at least, one terminal state). Such a solution would imply the attainment of riskless profit (arbitrage). For each scenario set that we used in the empirical analysis we numerically confirmed that its discrete distribution of asset prices indeed did not contain any “free lunch” opportunities.¹

Our primary motivation for this study is to develop suitable means for incorporating options in dynamic (multi-stage), scenario-based stochastic programming models for portfolio management. This would involve decision variables allowing transactions on options at each decision stage (i.e., at each node of the scenario tree). We present next two approaches to price options on a scenario tree. Both procedures can price European options at any node of the scenario tree, in accordance with the postulated discrete distribution of asset prices represented by the subtree emanating from the node of interest.

3 Pricing Options on a Scenario Tree

Say we want to price a European option with term (maturity) τ periods at a non-terminal node $n_0 \in \mathbf{N} \setminus \mathbf{N}_{\mathcal{T}}$ of the scenario tree. The underlying’s price at the time of the option’s valuation is that associated with node n_0 . The expiration date of the option coincides with a stage of the scenario tree. The essential input to price this option is the distribution of the underlying asset’s price at the option’s maturity, conditional on node n_0 . Consider the subtree with depth τ rooted at node n_0 (highlighted in Fig. 1). The leaf nodes, \mathbf{L}_{τ} , of this subtree represent the desired conditional distribution (i.e., the possible states of the underlying’s price at the option’s maturity date, conditional on the price being at state n_0 at the valuation date).

The discrete support of this distribution of the underlying’s price is $\Omega = \{\omega^n = S^n : n \in \mathbf{L}_{\tau}\}$. The corresponding conditional (on node n_0) probabilities of the physical distribution are $P = \{p_n = \pi_n / \pi_{n_0} : n \in \mathbf{L}_{\tau}\}$, where π_n was defined as the unconditional probability mass for node n .

In what follows, references to the underlying’s stochastic price at the option’s maturity, to the distribution of this price, to its discrete support or to the corresponding probabilities, always imply that the respective quantities are conditional on the underlying’s price being at node n_0 on the option’s valuation date.

We use the following notation:

- τ the term of the European option priced at state n_0 .
- S_0 the price of the underlying asset at node n_0 (i.e., at the root node of the subtree); this is deterministic for the associated option pricing case.
- \tilde{S}_{τ} the random price of the underlying asset at the expiration date of the option, conditional on the price S_0 at the option’s valuation date. This random variable takes values in the discrete set Ω .

¹In fact, we ran the tests for each subtree on which we carried out option pricing calculations. We ran the tests not only with the assets as the available investment outlets (decision variables of the linear programs), but also after incorporating the options — priced with the proposed procedures — in the optimization programs. In all cases we confirmed the absence of “free lunches” in the sets of asset price scenarios used in the numerical experiments.

- S^n the price of the underlying asset at node $n \in \mathbf{L}_\tau$ of the subtree. These prices and their respective probabilities over all leaf nodes, \mathbf{L}_τ , of the subtree represent the discrete, conditional distribution for the asset's price at the option's expiration date.
- r_f the riskless rate applicable during the lifetime of the option.
- K the exercise price of the option.
- P the physical probability measure for the discrete conditional distribution of the underlying's price at the option's maturity.
- \bar{P} an equivalent risk-neutral probability measure for the same discrete distribution.

We consider two approaches for pricing options on scenario trees of asset prices. In the first case we determine an equivalent risk-neutral probability measure over the discrete outcomes (S^n , $n \in \mathbf{L}_\tau$) of the asset's price. The risk-neutral probabilities are obtained through a transformation of the physical measure on these outcomes that applies a change of measure, while also satisfying the necessary martingale condition.

3.1 Method 1: Determining Risk-neutral Probabilities

In the no-arbitrage setting the price of a European option is computed as the expectation, with respect to a risk-neutral measure, of its discounted (by the riskless rate) payoff. Harisson and Kreps [17] applied this valuation principle and introduced the notion of a pricing functional which operates on the payoff of a contingent claim. Hence, we need to determine an equivalent risk-neutral probability measure for the discrete support, Ω , of the underlying's price at the option's expiration date.

A basic theorem in risk-neutral valuation (see Jacod and Shiryaev [21] for the theorem in discrete time) states that a model of asset prices is arbitrage-free iff there exists a probability measure \bar{P} (the risk-neutral measure) under which the discounted process of the asset prices is a martingale. The martingale condition requires that over a time interval the expected, under the risk-neutral measure \bar{P} , return of an asset is equal to the riskless return over the same interval (e.g., Neftci [29], chapter 15). That is,

$$E_{\bar{P}}[e^{-r_f \tau} \tilde{S}_\tau | S_0] = S_0. \quad (1)$$

Consequently, an equivalent martingale measure over the discrete support, Ω , of asset prices must satisfy the following system of linear equations and inequalities:

$$\sum_{n \in \mathbf{L}_\tau} \bar{p}_n S_i^n = S_{0i} (e^{r_f})^\tau, \quad i = 1, \dots, I, \quad (2)$$

$$\sum_{n \in \mathbf{L}_\tau} \bar{p}_n = 1, \quad (3)$$

$$\bar{p}_n > 0, \quad \forall n \in \mathbf{L}_\tau. \quad (4)$$

The equality (3) ensures that $\bar{P} = \{\bar{p}_n : n \in \mathbf{L}_\tau\}$ is a proper measure, and the inequalities (4) ensure the required equivalence between the physical, P , and the risk-neutral, \bar{P} , probability measures.

Equivalence between the two measures requires that they both assign nonnegative probability to the same domains (i.e., $P(Z) > 0 \Leftrightarrow \bar{P}(Z) > 0$, $P(Z) = 0 \Leftrightarrow \bar{P}(Z) = 0$, $\forall Z \subset \Omega$). In our setting, both measures are defined over the same discrete set, Ω , of asset price outcomes and (4) requires that the risk-neutral measure should assign positive mass, \bar{p}_n , to each atom of Ω (i.e., to each postulated conditional outcome of asset prices over the leaf nodes \mathbf{L}_τ of the subtree on which pricing is carried out). Harisson and Kreps [17] (Theorem 2) indicated that asset price models that admit no “free lunches” have an equivalent martingale measure. As we discussed in the previous section, we numerically verified that the asset price scenarios we use in the empirical tests of this study do not contain any “free lunches.” Consequently, there exists an equivalent risk-neutral (martingale) measure on the set of asset price scenarios.

In typical problem instances, the necessary martingale conditions (2)–(4) alone are not sufficient to determine the required risk-neutral probabilities because this linear system is usually underdetermined. For this system to be completely determined — so as to result in a unique martingale measure, — the number of linearly independent securities, I , must be equal to the number of price outcomes $|\mathbf{L}_\tau| - 1$. This is not usually the case in practical problem instances; typically, the number of price outcomes specified on a scenario tree to approximate the distribution of random prices is much larger than the number of assets considered in a problem.

Hence, we resort to developments in a market equilibrium approach to option pricing in order to supplement the necessary martingale conditions and determine the implied risk-neutral probabilities for the discrete price outcomes.

A common assumption in equilibrium valuation models is that the market participants can be aggregated into a representative agent. The utility function of the representative agent is a matter of choice among certain admissible classes. An appropriate choice is one that represents the aggregate market well; Jackwerth [19] suggests “a power utility function of moderate risk aversion.”

In equilibrium, the relationship between the physical and the risk-neutral probability measures through a pricing kernel leads to the following expression for the current price of an asset whose random value at time τ is \tilde{S}_τ :

$$S_0 = (e^{-r_f})^\tau E_P[\xi \tilde{S}_\tau], \quad (5)$$

where ξ is the pricing kernel (stochastic discount factor) and $E_P[\cdot]$ denotes the expectation operator with respect to the physical probability measure P . Explicit dynamic modelling of the joint stochastic process of asset returns and the pricing kernel can be found in the consumption-based equilibrium asset pricing literature (see, Aït-Sahalia and Lo [1], Jackwerth [19], Rosenberg and Engle [30] for applications to derivatives pricing).

Bakshi et al. [2] developed a method for relating the physical and risk-neutral probability measures through a pricing kernel. We follow their methodology, which we adapt to the case of a discrete distribution of asset returns. In a continuous setting, the equilibrium condition (5) relates the physical and the equivalent risk-neutral measure as follows:

$$S_0 = (e^{-r_f})^\tau E_P[\xi \tilde{S}_\tau] = (e^{-r_f})^\tau \int_{\Omega} \tilde{S}_\tau(\omega) \xi(\omega) dP(\omega) = (e^{-r_f})^\tau \int_{\Omega} \tilde{S}_\tau(\omega) d\bar{P}(\omega), \quad (6)$$

implying

$$\xi(\omega) dP(\omega) = d\bar{P}(\omega) \quad \Rightarrow \quad \frac{d\bar{P}(\omega)}{dP(\omega)} = \xi(\omega) \quad a.s. \quad (7)$$

The physical and the risk-neutral probability measures P and \bar{P} , respectively, are defined on a measurable space (Ω, F) .

This result takes a special form when the support of the asset price distribution is a finite discrete set, as is the case with the discrete outcomes $\Omega = \{\omega^n = S^n : n \in \mathbf{L}_\tau\}$, we use to represent the distribution of the stochastic asset price \tilde{S}_τ . In this case, F comprises all subsets of Ω . In the discrete distribution setting, the analogue of (7) is

$$\xi(\omega^n) = \frac{\bar{P}(\omega^n)}{P(\omega^n)} \Rightarrow \bar{P}(\omega^n) = \xi(\omega^n) P(\omega^n), \quad \forall \omega^n \in \Omega.$$

Bakshi et al. [2] derived a transformation between the physical and the risk-neutral probabilities. This transformation, specialized for the case of a discrete support, takes the form

$$\bar{p}_n = \frac{E_P[\xi | S^n] p_n}{\sum_{n \in \mathbf{L}_\tau} E_P[\xi | S^n] p_n}, \quad n \in \mathbf{L}_\tau, \quad (8)$$

where ξ is a general change-of-measure pricing kernel.

Under the common hypothesis of a power utility function of a representative market agent, the stochastic discount factor (pricing kernel) can be specialized to

$$E_P[\xi|S^n] = (S^n)^{-\gamma} = e^{-\gamma \ln(S^n)}, \quad n \in \mathbf{L}_\tau, \quad (9)$$

where γ is the coefficient of relative risk aversion.² Substituting (9) in (8) and dividing both the denominator and the numerator by $S_0^{-\gamma}$ we obtain

$$\bar{p}_n = \frac{e^{-\gamma \ln(S^n/S_0)} p_n}{\sum_{n \in \mathbf{L}_\tau} e^{-\gamma \ln(S^n/S_0)} p_n} = \frac{e^{-\gamma R^n} p_n}{\sum_{n \in \mathbf{L}_\tau} e^{-\gamma R^n} p_n}, \quad n \in \mathbf{L}_\tau, \quad (10)$$

where $R^n = \ln(S^n/S_0)$ is the return of the underlying asset at leaf node $n \in \mathbf{L}_\tau$, conditional on the initial price S_0 .

The transformation in (10) is intended to produce a risk-neutral measure, by “exponentially tilting” the physical measure, consistent with equilibrium principles. This transformation depends on a risk aversion parameter γ . We empirically estimate the value of this parameter using market prices of options and the underlying security.³

The estimated value of the parameter γ may be affected by measurement and estimation errors. We estimate the value of this parameter using observed option prices at a recent date, preceding the date of the analysis, and then use the same value to price options at stages of the scenario tree that correspond to later periods. The postulated conditional discrete distributions of asset returns at subsequent nodes of the scenario tree can, potentially, be different from the distribution that was used for the parameter estimation.

Hence, we explicitly impose the necessary martingale conditions in the determination of the risk-neutral probabilities before carrying out option pricing computations at a node $n_0 \in \mathbf{N} \setminus \mathbf{N}_\tau$ of the scenario tree. We obtain the (conditional on state n_0) risk-neutral probabilities of the price states $n \in \mathbf{L}_\tau$ at the option’s expiration date from the solution of the quadratic program:

$$\begin{aligned} \text{minimize}_{\bar{p}_n \in \mathbf{L}_\tau} \quad & \sum_{n \in \mathbf{L}_\tau} (\bar{p}_n - \hat{p}_n)^2 \\ \text{s.t.} \quad & \sum_{n \in \mathbf{L}_\tau} \bar{p}_n S^n = S_0 (e^{rf})^\tau \\ & \sum_{n \in \mathbf{L}_\tau} \bar{p}_n = 1 \\ & \bar{p}_n > 0, \quad \forall n \in \mathbf{L}_\tau \end{aligned} \quad (11a)$$

The values \hat{p}_n are computed by the transformation (10) and reflect conditional probabilities for the discrete states $n \in \mathbf{L}_\tau$ implied by equilibrium principles. The risk-neutral (martingale) measure

²Equations (8) and (9) are developed in Bakshi et al. [2]; see also references therein citing applications of the same constructs.

³Let,

MP_i the observed market price of option $i = 1, \dots, m$, and

$CP_i(\gamma)$ the price of option $i = 1, \dots, m$, computed with Method 1 as discussed in this section; this price results from the pricing procedure using a coefficient of relative risk aversion γ .

We estimate the value of the parameter γ so as to minimize the sum of the relative squared pricing errors over a set of options $i = 1, \dots, m$, with the same maturity (differing in strike price). That is, we (approximately) solve with a directed search the following unconstrained quadratic program:

$$\gamma = \operatorname{argmin}_{\gamma'} \sum_{i=1}^m ((CP_i(\gamma') - MP_i)/MP_i)^2.$$

$\bar{P} = \{\bar{p}_n : n \in \mathbf{L}_\tau\}$ obtained from the solution of (11) is the closest (in a Euclidean sense) to the measure, \hat{P} , implied by equilibrium principles.

A similar idea is used in de Lange et al. [8]. They also minimize a metric between prices that satisfy the martingale requirement and reasonable equilibrium prices. Rubinstein [32] used an analogous idea. He also determined a risk-neutral measure by minimizing its distance from a reference distribution. First, he established the risk-neutral probabilities in a standard binomial tree. Then he determined the implied posterior risk-neutral probabilities using a quadratic program. These probabilities were, in the least square sense, those closest to log-normal that caused the present value of the underlying asset and all options calculated with these probabilities to fall between their respective bid and ask prices.

Once we determine the equivalent risk-neutral probabilities \bar{p}_n for the discrete outcomes, S^n , $n \in \mathbf{L}_\tau$, of the underlying asset's price, pricing the options is straightforward. The fair price of a European call option at node n_0 , with strike price K , is the expected value of its payoff over the risk-neutral measure, discounted by the riskless rate:

$$c_0(S_0, K) = (e^{-r_f})^\tau \sum_{n \in \mathbf{L}_\tau} \bar{p}_n [\max(S^n - K, 0)], \quad (12)$$

The fair price of a European put option with strike price K is similarly computed by:

$$p_0(S_0, K) = (e^{-r_f})^\tau \sum_{n \in \mathbf{L}_\tau} \bar{p}_n [\max(K - S^n, 0)]. \quad (13)$$

The transformations in (10), and the pricing calculations (12) and (13), are straightforward. But the quadratic program (11) must be solved at each node where an option valuation is to be carried out. Most involved is the procedure for estimating the risk aversion parameter γ (summarized in footnote 3) because the function that is minimized is only implicitly expressed in terms of the variable. The estimation of the parameter γ is done once, and then the estimated value is used in all subsequent option valuations.

The advantage of this approach lies in its generality. It is, in principle, applicable for any discrete representation (scenario tree) of the random asset prices, and values the options by explicitly considering these discrete distributions.

King [26] develops a mathematical basis for analyzing contingent claims in the discrete time, discrete state case. He models the hedging problem as a stochastic program. He shows that the absence of arbitrage (i.e., boundedness) in the hedging problem translates in the dual to the existence (feasibility) of a valuation operator (probability measure) that makes the discrete discounted price process into a martingale. In complete markets the dual problem establishes the fundamental asset pricing theorems and determines the unique valuation operator (equivalent risk-neutral martingale measure). In incomplete markets the dual has multiple solutions giving rise to a range of prices for the contingent claim. Bid and ask prices for the claim are characterized by the extremal cases of the dual's feasibility set. A similar approach for contingent claims analysis is summarized in Zhao and Ziemba [40].

In complete markets both King's approach and the method described above will determine the same (unique) martingale measure over the discrete set of price outcomes, and hence the same prices for options on the constituent assets. In this case we will not have to solve the quadratic program (11); when there is a sufficient number of linearly independent securities so that the system corresponding to the martingale conditions (2)–(4) is completely determined the required martingale measure is thus obtained from its unique solution. However, for the case of incomplete markets we resort to market equilibrium principles in order to determine a point estimate (approximation) of the option's price — associated with a specific choice of an equivalent martingale measure — rather than a price range within a bid-ask interval.

3.2 Method 2: Accounting for Skewness and Kurtosis of Asset Returns

We explore another approach that prices options on the basis of an empirical distribution of asset prices. The theoretical foundation for this method was introduced by Jarrow and Rudd [24] and further developed by Corrado and Su [7] (Brown and Robinson [6] provided a correction note).

Let,

- τ the term of a European option,
- S_0 the market price of the underlying asset at the time of the option's valuation,
- \tilde{S}_τ the random price of the underlying asset at the option's expiration date,
- r_f the riskless rate,
- K the exercise price of the option.

The log-return of the asset during the holding period is:

$$\tilde{r}_\tau = \ln \tilde{S}_\tau - \ln S_0 = \ln(\tilde{S}_\tau/S_0). \quad (14)$$

Then

$$\tilde{S}_\tau = S_0 e^{\tilde{r}_\tau}, \quad (15)$$

and the conditional distribution of \tilde{S}_τ depends on that of the log-return \tilde{r}_τ .

In the risk-neutral setting, the price, c_0 , of a call option is computed as:

$$c_0 = (e^{-r_f \tau})^\tau E[(\tilde{S}_\tau - K)^+] = (e^{-r_f \tau})^\tau \int_{\ln(K/S_0)}^{\infty} (S_0 e^x - K) f(x) dx, \quad (16)$$

where $f(\cdot)$ is the conditional density of \tilde{r}_τ .

Corrado and Su [7] applied a Gram-Charlier series expansion to represent the empirical probability density of asset log-returns, in order to derive an option pricing formula. The series expansion approximates the underlying distribution with an alternate (more tractable) distribution, specifically, with the log-normal. The coefficients in the expansion are functions of the moments of the original and the approximating distributions.

A truncation of the series to only two terms results in the familiar Black-Scholes pricing formula. Additional terms in the series expansion account for the effect of higher order moments of the asset returns' distribution on the option price. Corrado and Su derived the pricing formula based on a four-term Gram-Charlier series expansion so as to account for skewness and kurtosis of asset returns. Intuition suggests that for practical purposes the first four moments of the underlying distribution should capture the majority of its influence as it affects option pricing (see, e.g., Jarrow and Rudd [24], p. 349).

Using a four-term Gram-Charlier series expansion for the conditional density $f(\cdot)$, Corrado and Su solved (16) and obtained the following expression for the price of a call option:

$$c_0 = C_{BS} + \gamma_1 Q_3 + (\gamma_2 - 3) Q_4, \quad (17)$$

where C_{BS} is the Black-Scholes price, and Q_3 and Q_4 represent adjustments for nonzero skewness and kurtosis respectively. The quantities in (17) are computed as follows:⁴

$$C_{BS} = S_0 N(d) - K e^{-r_f \tau} N(d - \sigma_\tau), \quad (18)$$

$$Q_3 = \frac{1}{3!} S_0 \sigma_\tau [(2\sigma_\tau - d)\varphi(d) + \sigma_\tau^2 N(d)], \quad (19)$$

$$Q_4 = \frac{1}{4!} S_0 \sigma_\tau [(d^2 - 3d\sigma_\tau + 3\sigma_\tau^2 - 1)\varphi(d) + \sigma_\tau^3 N(d)], \quad (20)$$

$$d = \frac{\ln(S_0/K) + r_f \tau + \sigma_\tau^2/2}{\sigma_\tau}. \quad (21)$$

⁴ $\varphi(\cdot)$ is the standard normal density, $N(\cdot)$ is the cumulative normal distribution, $\gamma_1 = \mu_3/\mu_2^{(3/2)}$ and $\gamma_2 = \mu_4/\mu_2^2$ are the Fisher parameters for skewness and kurtosis, respectively, and μ_i is the i^{th} central moment of asset returns.

The price, p_0 , of a put option with the same strike price, K , is determined by put-call parity:

$$p_0 = c_0 + Ke^{-rf\tau} - S_0. \quad (22)$$

This method augments the B-S price with additional adjustments that account for the effects of skewness and kurtosis of the distribution of asset returns. It is simple to implement and is computationally efficient. Unlike the previous approach, it does not require the solution of an optimization program or other involved computations. It involves only fairly simple calculations and can be readily applied to price options at any node of the scenario tree. Besides basic deterministic inputs (r_f , S_0 , K) it only requires estimates of the moments of the underlying asset's return during the holding period. These moments are determined on the basis of the discrete outcomes of the asset prices at the option's maturity, as specified at the leaf nodes $n \in \mathbf{L}_\tau$ of the subtree on which pricing is carried out (as shown in Figure 1).

With the moment-matching procedure that we employed in this study, the discrete distributions are generated so that the moments of asset returns match corresponding empirical estimates that are determined on the basis of observed market data. Although this pricing procedure does not directly use the specific outcomes prescribed on the scenario tree, it conforms to the discrete distribution they represent as it uses the same statistics (first four moments) of asset returns.

4 Empirical Features

4.1 Effects of Higher Moments

The moment-matching method allows full control of the moments when generating scenarios. We used this capability to numerically investigate and quantify the sensitivity of option prices to higher moments (skewness and kurtosis) of the distribution of the underlying's returns, and to examine the extent to which option prices computed with the proposed procedures differ from B-S prices in cases of asymmetric and heavy-tailed distributions of asset returns.

We carried out tests using market data of the *S&P500* index as of November 20, 1999. The index price on that date was $S_0 = 1424.26$. From market values of the index over the period November 1989 to October 1999 we computed the following moments of monthly log returns: mean=0.007356, variance=0.018762, skewness=0.011508, excess kurtosis=0.029589. We first generated 15,000 scenarios of index prices matching exactly these empirical moments. We generated additional scenario sets (with 15,000 scenarios each) by varying only the skewness, or only the kurtosis, in the range of $\pm 25\%$ in increments of 5%, relative to their empirical values. We applied the two pricing procedures described in the previous section to compute option prices based on these scenario sets.

We considered two different call options. One with a strike price $K = 0.95S_0$ (in-the-money, ITM) and one with a strike price $K = 1.05S_0$ (out-of-the-money, OTM). From the results of the tests, shown in Figures 2 and 3, we observe the following:

1. The two approaches consistently produce very similar option prices in all cases.
2. For the ITM option (Fig. 2), both procedures result in prices that are very close to the Black-Scholes price when the empirical moments of the index return are used. This is because the reference (empirically estimated) distribution of asset returns in this test does not deviate very significantly from the log-normal. Moreover, the price of the ITM option is not (proportionally) very sensitive to small variations of the higher moments. However, the Black-Scholes formula clearly overprices the OTM call option (Fig. 3).
3. Both approaches exhibit consistent behavior in terms of the sensitivity of option prices to changes in the higher moments of the index' returns; the observed effects on their computed option prices follow the same patterns. Of course the Black-Scholes price is unaffected by variations of higher moments, as it depends only on the first two moments.

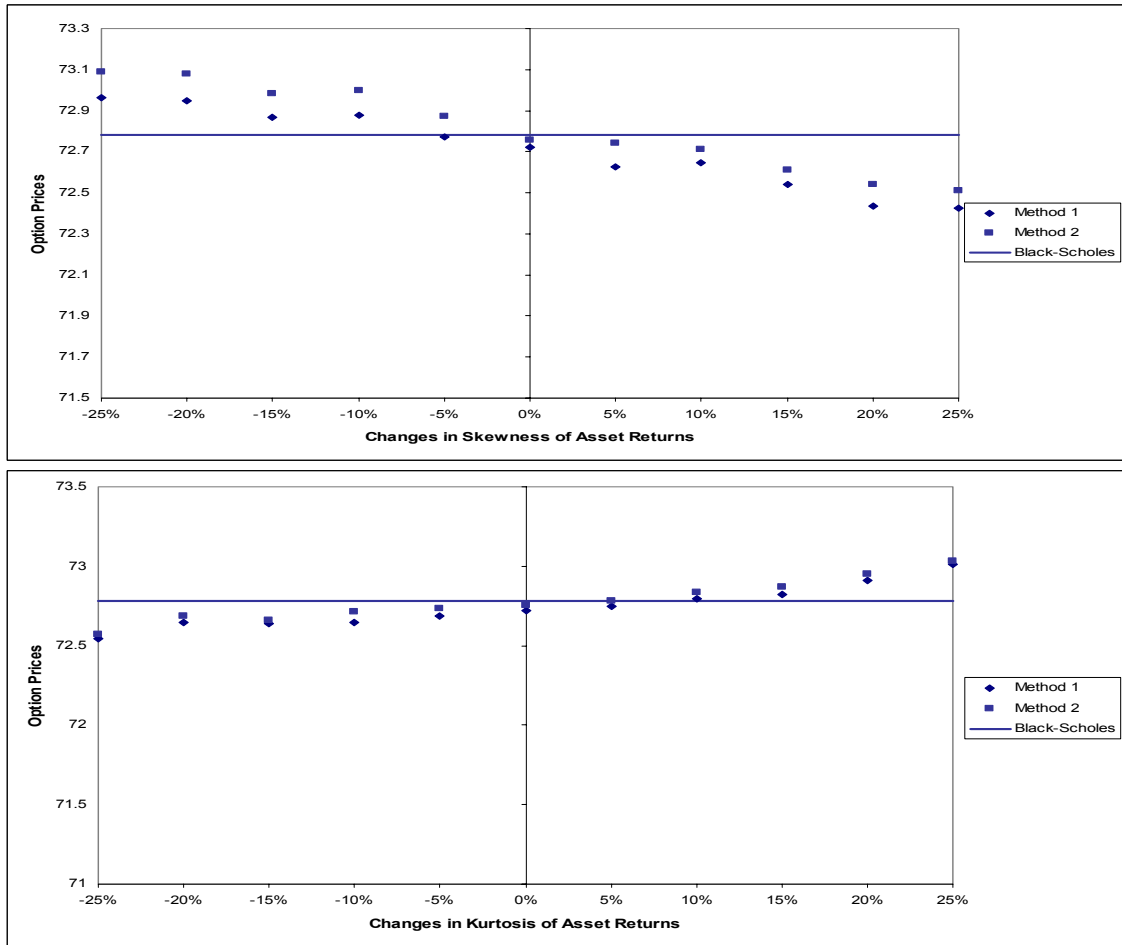


Figure 2: Effects of changing skewness and kurtosis of asset returns on ITM call option prices.

4. As expected, the ITM call option is much more expensive than the OTM.

For the ITM call option, a decrease in skewness of the underlying’s return — especially negative skewness — causes an increase in the price of the option, while an increase in skewness results in lower option prices. The opposite trend is observed for changes in the kurtosis of the underlying index’ returns. Changes in the higher moments did not have substantial (proportionally) impact on ITM call prices. Of course this sensitivity of ITM call option prices to higher moments depends on the level of the induced changes and on the degree of the option’s “moneyness”.

More frequently interesting in practice are OTM options, whether call or put, which are cheaper than ITM options. Long positions in OTM put options provide downside risk coverage, while long positions in OTM call options enhance upside potential.

For the OTM call option, the computed price exhibits an increasing trend in terms of both the skewness and the kurtosis of the underlying index’ returns. Except for the high values of kurtosis or/and skewness, the Black-Scholes formula overprices the OTM call option in comparison to the proposed procedures. This is an empirically observed limitation of the Black-Scholes formula (i.e., it underprices deep OTM put options and overprices deep OTM call options). We also observe a higher (proportional) sensitivity of OTM call option prices to changes in the higher moments of asset returns, in comparison to ITM call options.

The results illustrate that the proposed pricing procedures are more flexible, and more consistent in estimating market prices of options, in comparison to the Black-Scholes approach. This is a consequence of the fact that they account for the effect of higher moments of the underlying’s

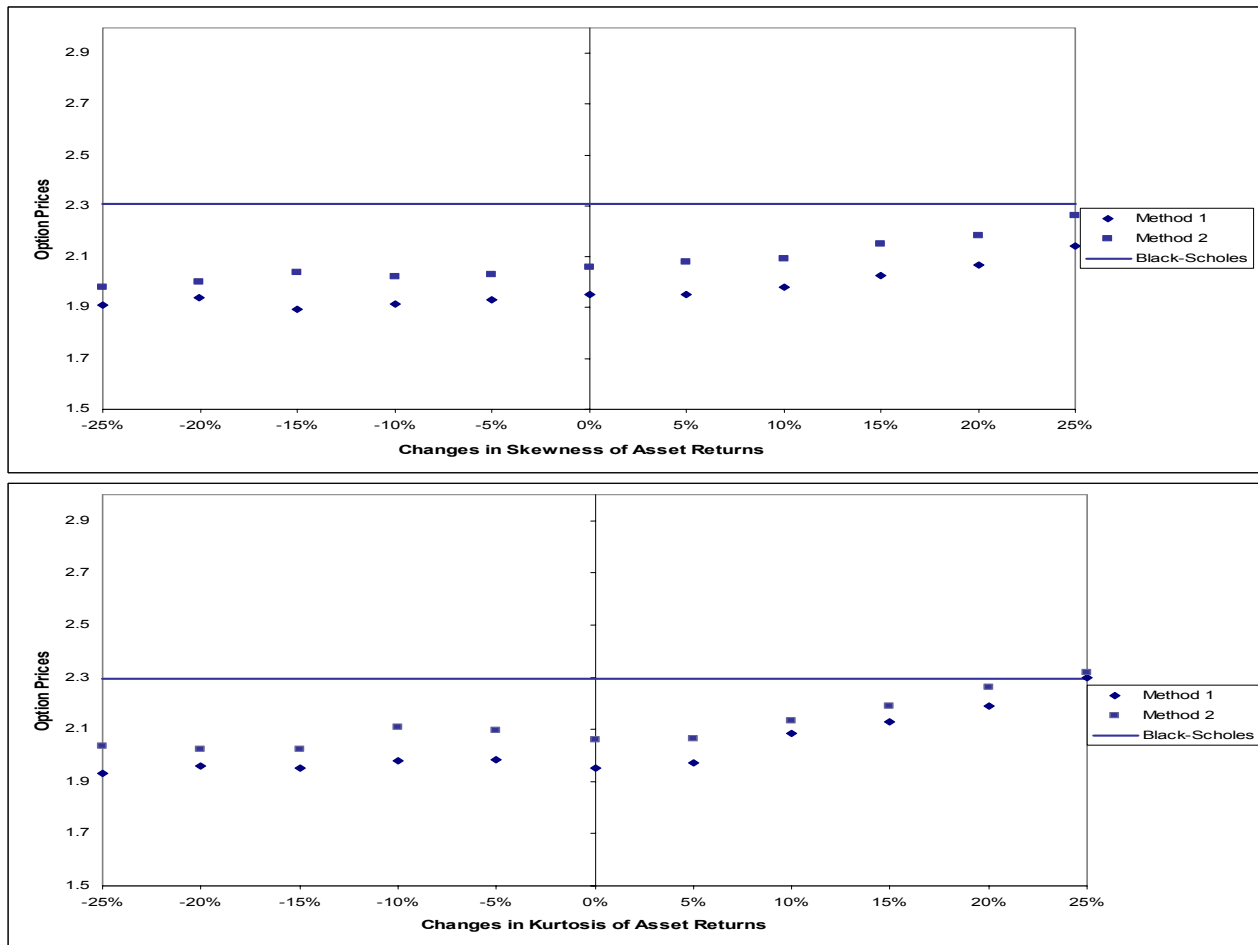


Figure 3: Effects of changing skewness and kurtosis of asset returns on OTM call option prices.

returns and can react to changes of these statistics, while the B-S formula accounts only for the first two moments. These results are qualitatively consistent with those reported by Corrado and Su [7] and Theodosiou and Trigeorgis [33] on the effect of higher order moments of asset returns on option prices; Theodosiou and Trigeorgis employ a range of different distributions in their analysis.

4.2 Implied Risk Neutral Distributions and Skewness Premia

The insight from equations (12) and (13) is that the price of OTM calls is most affected by the upper tail of the risk-neutral distribution, while the price of OTM puts is most affected by the lower tail. This is because an OTM call yields payoffs only if the asset price rises above the exercise price, while an OTM put yields payoffs only when the asset price falls below the strike price. As Bates [3] notes, the symmetry or asymmetry of the risk-neutral distribution is reflected in the relative prices of OTM calls and puts. Symmetric risk-neutral distributions imply equal prices for symmetric OTM calls and puts (i.e., options whose exercise prices are spaced symmetrically around the mean asset price ($S_0 e^{rf\tau}$) at the expiration date).

A skewness measure of the risk-neutral distribution is the “skewness premium”, defined by Bates [3] as the percentage deviation of $a\%$ OTM call prices from $a\%$ OTM put prices:

$$SP(a) = \frac{c_0(S_0, K_c)}{p_0(S_0, K_p)} - 1 \quad (23)$$

where $K_c = (1 + a)S_0 e^{rf\tau}$ and $K_p = (1 - a)S_0 e^{rf\tau}$ are the strike prices of the $a\%$ -symmetric OTM

call and put option, respectively.

Method 1 allows to estimate directly a discrete approximation of the risk-neutral distribution of asset returns. Using this approach, we estimate the risk-neutral distribution of monthly returns for the *S&P500* index at each month during the period of Jan. 1999 to Nov. 2002. Based on the estimated distributions, we calculate the skewness premia for 5% and 2% OTM call and put options, according to equation (23). The resulting skewness premia — from computed option prices — are plotted in Figure 4 and show that the implied risk-neutral distributions for log-returns of the *S&P500* index during this period are not symmetric. Three distinct regimes can be noticed. From Jan.-99 to Sep.-00, OTM puts are more expensive than OTM calls, yielding negative skewness premia. Positive skewness premia are observed during the short period between Oct.-00 and Feb.-01, while from Mar.-01 until Nov.-02 the skewness premia are again negative. Figure 4 also confirms that the deeper OTM the options are, the higher, in absolute value, the respective skewness premium.

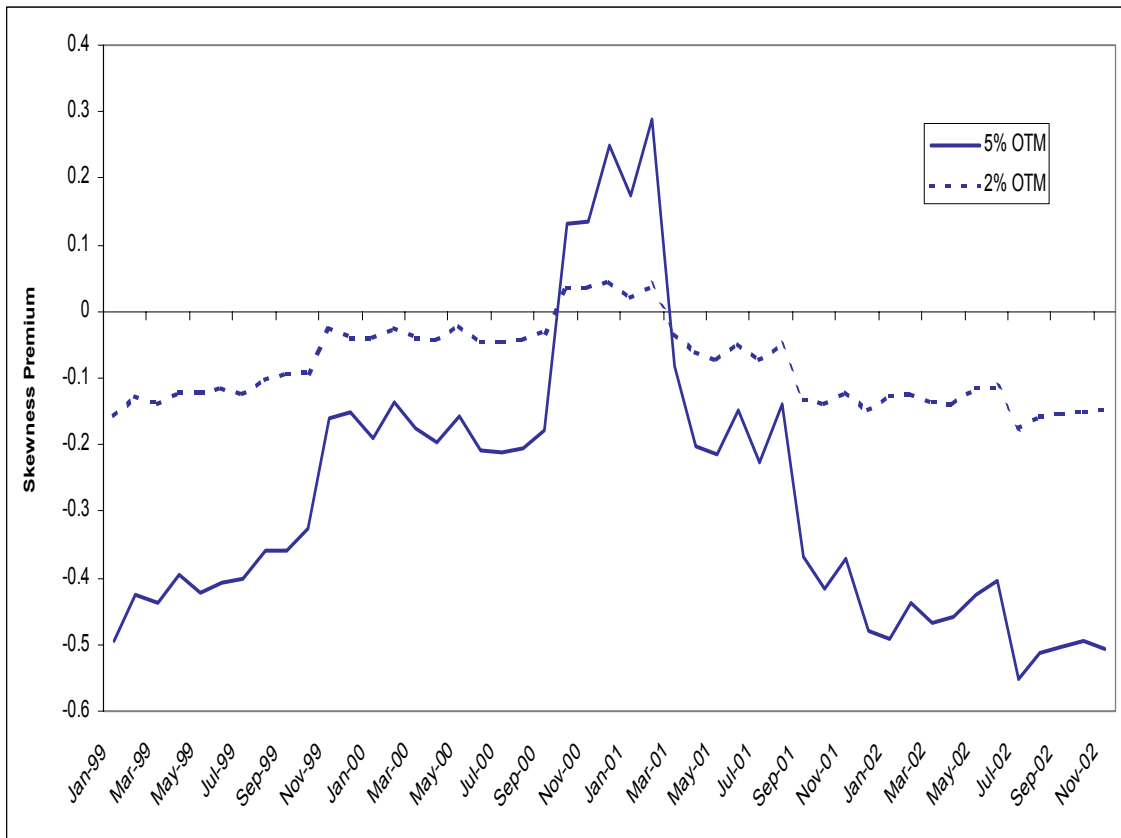


Figure 4: Empirical estimates of skewness premia for the *S&P500* index.

Skewness premia provide an indication of market expectations. If skewness premia are negative, the market considers a decrease in the value of the underlying more likely than an increase. This would imply a negatively skewed distribution, and a higher probability that an OTM put option will be exercised at maturity compared to the corresponding probability that a symmetric OTM call option will be exercised.

Figure 5 plots the *S&P500* index between October 1988 and December 2002. Comparing Figures 4 and 5 we observe that skewness premia varied in a similar pattern to the underlying — the *S&P500* index. They exhibit an upward trend during periods of market advancements, while they take increasingly negative values during periods of market declines. The skewness premia are positive during the period in which the *S&P500* had peaked.

Figure 6 plots the estimated risk-neutral distributions of the *S&P500* index' returns at two points

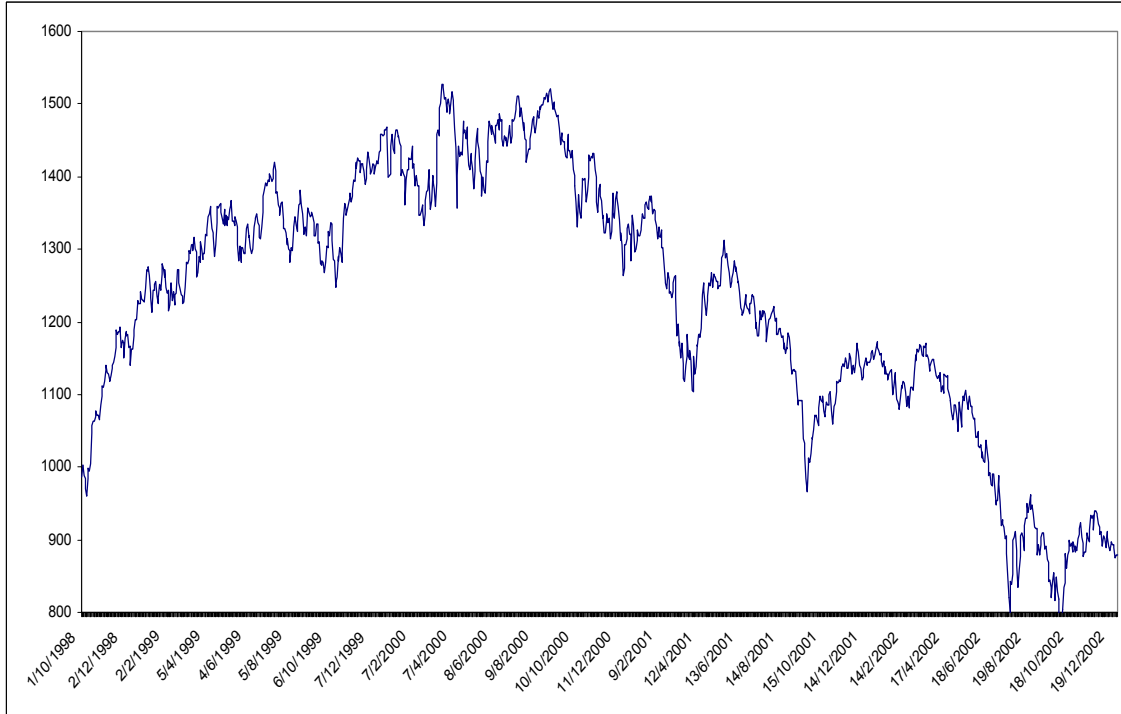


Figure 5: Market values of the *S&P500* index.

in time; the first when the skewness premium was positive (Dec. 2000) and the second when it was negative (June 2002). The value of the index, S_0 , at the corresponding date and the strike prices K_c and K_p for $a = 5\%$ OTM call and put options, respectively, are also marked on the graphs.

We see that the estimated risk neutral distribution on Dec. 2000 (first graph) is positively skewed. The probability that the index will exceed the strike price, K_c , of the OTM call option is higher than the probability that it will fall below the strike price, K_p , of the symmetric OTM put option. This explains why the OTM call is more expensive than the symmetric OTM put in this case.

The assumption of a geometric Brownian motion for asset prices in the Black-Scholes framework implies a log-normal risk-neutral distribution (i.e., positively skewed). When the empirically estimated risk-neutral distribution is indeed positively skewed — as is the case for the *S&P500* index returns on 12/2000 (first graph of Fig. 6) — then the two proposed procedures produce prices for OTM call and put options that are close to the Black-Scholes prices. Their price differences depend only on the degree that the empirically determined risk-neutral distribution differs from the log-normal.

This does not hold when the estimated risk-neutral probability distribution for asset returns comes out to be negatively skewed, as is the case for the *S&P500* index returns on June 2002 (second graph in Figure 6). The probability that the index will fall below the strike price, K_p , of the OTM put option is now higher than the probability that it will exceed the strike price, K_c , of the symmetric OTM call option. Hence, the OTM put becomes relatively more expensive than the respective OTM call, which is reflected in the negative skewness premium.

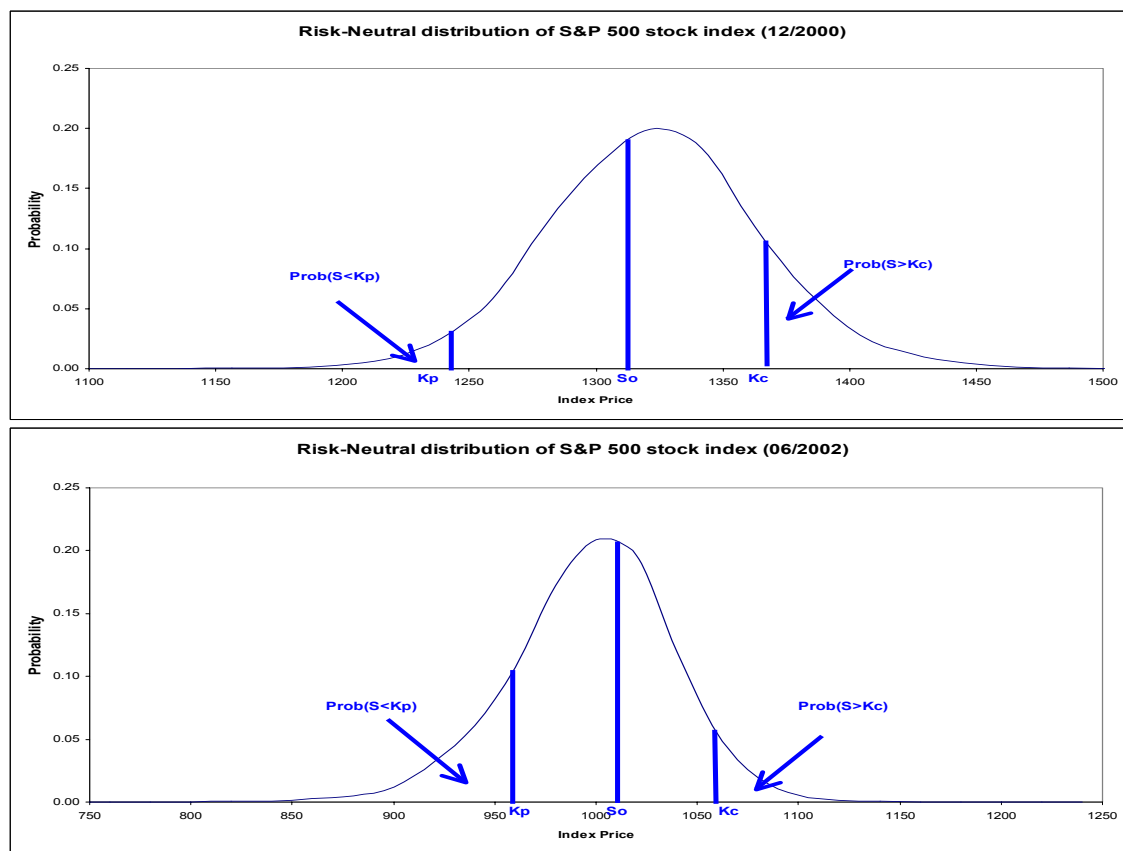


Figure 6: Estimated risk-neutral distributions of the *S&P*500 index returns. The distribution is positively skewed in Dec. 2000 and negatively skewed in June 2002.

5 Empirical Validation of Methods using Market Option Prices

We empirically assess the performance of the proposed option-pricing procedures. We compare option prices obtained with the proposed valuation procedures, and with the Black-Scholes method, to market prices of options on the S&P500 stock index. The B-S prices serve as reference point for comparison. Empirical evidence of a valuation method's ability to approximate market prices of options, without systematic biases, provides a proper basis to assess its efficacy. Keeping in mind, however, that no valuation approach reproduces exactly the option prices observed in the markets. Justifications for the differences between model and market prices of options are pointed out by Bates [4], Jarrow and Madan [23], and Jacquier and Jarrow [22].

A critical parameter of the Black-Scholes pricing formula is the instantaneous volatility, σ , of the underlying's return. One can either use an estimate of volatility based on historical observations of asset prices, or can compute an estimate of volatility that is implied by quoted option prices at a particular time of interest. We used both approaches in our tests.

We considered European call options on the S&P500 stock index with maturity, τ , of one month. All data for the analysis were obtained from Datastream. We priced European call options at two different dates: December 20, 2000 and June 20, 2002. Our estimates of the skewness premia at these two dates have opposite sign, implying different market regimes. Basic data for the analysis are summarized in Table 1.

We see that the skewness premia implied by market prices of options confirm the corresponding premia determined from estimates of the risk neutral distribution of the S&P500 index for the

	Dec. 20, 2000	June 20, 2002
Value of S&P500 stock index (S_0)	1305.6	1006.29
Monthly riskless interest rate (r_f)	0.06578%	0.018762%
Skewness Premia from Estimated Risk Neutral Distributions		
2% OTM	0.044	-0.114
5% OTM	0.247	-0.403
Skewness Premia Implied by Market Prices of Options		
~3% OTM	0.082	-0.1853
Estimated Statistics of Monthly Returns of the S&P500		
Mean (μ_1)	0.01208	0.0073
Standard Deviation (μ_2)	0.1826	0.1926
Skewness (μ_3)	-0.00344	-0.115
Kurtosis (μ_4)	0.025	0.029
Implied Volatility (σ_{ISD})	0.1909	0.1969

Table 1: Basic data for option valuation on 20/12/2000 and 20/06/2002.

respective dates.⁵ A negative skewness premium reflects a negatively skewed risk-neutral distribution for the underlying. In such a case, the Black-Scholes approach systematically overprices OTM calls and underprices OTM puts; the mispricing is higher for deep OTM options. We show that this is indeed observed with market prices of options.

Tables 2 and 3 compare the performance of the valuation procedures, and contrast their estimated option prices against market prices of the options on 20/12/2000 and 20/06/2002, respectively.⁶ A number of one-month European call options on the S&P500 stock index with different strike prices are considered. Call options with strike prices lower, respectively higher, than the price of the S&P500 index, S_0 , at the respective date (i.e., ITM and OTM call options, respectively) are separated by a dividing line in each table.

The valuation methods include the two procedures described in section 3, and two variants of the Black-Scholes approach: B- S_h refers to the B-S method using an estimate of volatility σ from historical observations of the index, while B- S_I refers to the B-S method using the volatility estimate σ_{ISD} implied by the entire set of option quotations on the respective date.

In all cases the two proposed procedures consistently yield option prices that are quite close, even though the manner in which they account for the effect of higher order moments of the underlying is different. Their option price estimates approximate more closely than the Black-Scholes model the market prices of deep ITM, and especially of OTM, options. Black-Scholes prices have slightly smaller errors only for a few ITM call options. While the Black-Scholes approach substantially overprices deep OTM call options — and would correspondingly underprice deep OTM put options, — the proposed procedures approximate more closely the market prices of these options. OTM puts are particularly suitable for controlling downside risk and are also inexpensive. The systematic mispricing of OTM options by the Black-Scholes method is particularly evident when the skewness premium is negative, i.e., in June 2002 (see, Table 3).

⁵The skewness premia from observed market prices of options are approximate, as there were no price quotations for exactly symmetric call and put options (i.e., same OTM level). We used the available price quotations for options whose strike prices were closest to 3% OTM. The data confirm the different sign of the skewness premia at the two dates.

⁶A discrete distribution with 15,000 scenarios of index prices at the options' expiration date is used for pricing each option with the proposed procedures. In each case, the index' price scenarios are generated with the moment-matching method so that the first four moments of the index' returns match their corresponding empirical estimates, computed on the basis of observed index values during the preceding ten years.

Strike Price	Market Price	Estimated Option Prices							
		Method 1		Method 2		$B - S_h$		$B - S_I$	
		Price	Error	Price	Error	Price	Error	Price	Error
950	354.5	360.2	1.61%	360.4	1.66%	360.8	1.78%	360.8	1.78%
975	329.9	335.3	1.64%	335.1	1.58%	335.9	1.84%	335.9	1.84%
995	310.1	314.7	1.48%	315.0	1.57%	316.0	1.91%	316.0	1.91%
1025	291.8	286.4	-1.83%	285.8	-2.04%	286.2	-1.90%	286.2	-1.90%
1050	267.0	261.3	-2.13%	261.1	-2.21%	261.3	-2.12%	261.3	-2.12%
1075	242.4	236.4	-2.47%	236.5	-2.42%	236.5	-2.43%	236.5	-2.43%
1100	216.3	211.2	-2.35%	210.8	-2.53%	211.6	-2.15%	211.6	-2.15%
1125	192.9	186.4	-3.36%	185.9	-3.62%	186.8	-3.16%	186.8	-3.15%
1150	168.1	161.1	-4.18%	161.0	-4.24%	162.0	-3.64%	162.1	-3.60%
1200	117.9	111.4	-5.49%	111.2	-5.66%	113.4	-3.83%	113.7	-3.58%
1250	58.5	63.1	7.86%	63.0	7.69%	68.8	17.65%	69.6	19.03%
1270	45.4	48.0	5.79%	48.3	6.45%	53.5	17.80%	54.5	20.03%
1275	42.4	45.8	8.08%	46.1	8.79%	49.9	17.76%	51.0	20.26%
1285	36.0	38.1	5.83%	38.4	6.67%	43.2	19.98%	44.3	23.14%
1300	28.0	30.2	7.86%	30.4	8.57%	34.2	21.98%	35.4	26.33%
1320	21.0	22.2	5.71%	22.5	7.14%	24.1	14.76%	25.3	20.67%
1325	18.8	19.6	4.53%	19.9	6.13%	21.9	17.02%	23.2	23.59%
1345	13.1	14.3	8.95%	14.4	9.71%	14.7	11.76%	15.8	20.43%
1350	11.4	12.4	8.92%	12.5	9.89%	13.1	15.79%	14.3	25.48%
Mean Absolute Error			4.37%		4.82%		9.24%		11.69%
RMSE			5.40%		5.93%		12.11%		15.42%

Table 2: Observed vs estimated prices of European call options on the S&P500 index (December 20, 2000).

6 Conclusions

The objective of this work was not the pursuit of novel theoretical developments in option pricing. Instead, its contribution lies in the adaptation and empirical assessment of appropriate valuation methods to price options in a scenario-based framework so as to incorporate them in stochastic programming models for portfolio optimization. This is motivated by the fact that options constitute particularly suitable instruments for risk management purposes due to the asymmetric form of their payoffs. The capability to incorporate options in stochastic programming models for dynamic portfolio management problems enhances the potential applicability of such models, especially for applications in which risk management is a prime concern.

Stochastic programs have the flexibility to accommodate general discrete distributions in terms of scenario sets that do not necessarily conform to distributional assumptions of common option pricing methods. This necessitates the adaptation of suitable procedures for pricing options in accordance with discrete distributions of asset prices represented by scenario trees, so as to ensure internal consistency of the stochastic portfolio optimization models.

We explored and implemented two procedures to price options in accordance with discrete representations of stochastic asset prices. The proposed option valuation procedures take into account higher order moments of the empirical distribution of the underlying asset's returns. We generated scenarios of asset returns with a moment-matching procedure so as to capture the empirical moments. We numerically verified that the asset price scenarios are arbitrage-free. We showed that the higher order moments clearly affect the estimated prices of options; their effects can explain, at least

Strike Price	Market Price	Estimated Option Prices							
		Method 1		Method 2		$B - S_h$		$B - S_I$	
		Price	Error	Price	Error	Price	Error	Price	Error
800	207.7	207.9	0.10%	207.6	-0.02%	207.6	-0.04%	207.5	-0.09%
825	183.2	182.8	-0.25%	182.6	-0.33%	182.6	-0.30%	182.5	-0.36%
850	159.0	157.6	-0.87%	157.6	-0.90%	157.7	-0.82%	157.6	-0.88%
875	135.2	132.7	-1.83%	132.6	-1.94%	132.8	-1.79%	132.7	-1.83%
900	112.9	107.9	-4.42%	107.5	-4.82%	108.0	-4.30%	108.1	-4.23%
925	89.4	83.4	-6.74%	82.3	-7.98%	83.9	-6.15%	84.3	-5.74%
930	72.5	77.5	6.96%	77.3	6.60%	79.2	9.26%	79.7	9.88%
935	67.6	72.7	7.61%	72.4	7.06%	73.1	8.18%	75.1	11.12%
950	54.2	57.7	6.40%	58.2	7.36%	61.2	12.95%	62.0	14.44%
960	48.6	48.6	0.07%	49.4	1.71%	52.8	8.70%	53.8	10.77%
965	43.2	45.3	4.78%	45.3	4.90%	48.8	13.02%	49.9	15.75%
975	37.0	37.7	1.86%	37.9	2.34%	41.3	11.51%	42.5	14.97%
990	27.2	26.7	-2.00%	27.4	0.87%	31.1	14.46%	32.6	19.94%
995	24.3	23.7	-2.60%	24.4	0.49%	28.1	15.70%	29.7	22.03%
1005	20.3	19.9	-1.77%	20.4	0.72%	22.6	11.49%	24.2	19.35%
1010	17.5	17.8	1.74%	17.9	2.55%	20.1	14.99%	21.8	24.44%
1020	14.2	15.0	5.51%	15.3	7.52%	15.8	11.34%	17.4	22.48%
1025	12.1	13.1	8.52%	13.3	9.64%	15.0	24.08%	15.5	27.70%
1030	10.6	11.6	9.77%	11.7	10.79%	13.2	24.94%	13.7	28.99%
1040	7.6	8.6	13.25%	8.7	14.90%	10.2	33.90%	10.6	39.17%
1050	6.3	6.7	6.83%	6.8	8.62%	7.7	22.00%	8.0	27.77%
1060	4.2	4.7	12.09%	4.8	13.76%	5.7	35.83%	6.0	43.47%
Mean Absolute Error			4.74%		4.82%		8.56%		11.04%
RMSE			6.13%		6.85%		16.26%		20.49%

Table 3: Observed vs estimated prices of European call options on the S&P500 index (June 20, 2002).

partly, the difference between Black-Scholes prices and the market prices of options.

We demonstrated through extensive numerical tests using real market data that the proposed valuation procedures produce estimates of option prices that are consistently similar, and approximate more closely than the Black-Scholes method market quotations of option prices. They also exhibit consistent behavior in terms of the sensitivity of their computed option prices to changes in the higher moments of asset returns. The numerical results show that the proposed valuation procedures are notably more effective in those cases that the Black-Scholes formula is known to systematically misprice options (i.e., overprice OTM call options and underprice OTM put options, especially when the skewness premium is negative). Such conditions are common in practice (e.g., see Figure 3).

The numerical results show that the pricing of European options based on scenario trees that capture asymmetries and heavy tails of empirical asset return distributions is a viable valuation alternative. As we discussed in section 3, the proposed procedures can be applied to price European options at any decision node of a scenario tree. Hence, these procedures can provide flexible tools to incorporate options in stochastic programming models.

In [34, 35] we employed the procedures suggested in this paper to price options on stock indices denominated in different currencies, as well as quanto options on these indices, and incorporated these options in stochastic programming models for international portfolio management. In numerical

tests using market data we found that the inclusion of the options improved the performance of international portfolios by reducing risk while maintaining upside potential.

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