EXPERT OPINION ELICITATION IN OPTION PRICING: A BAYESIAN APPROACH
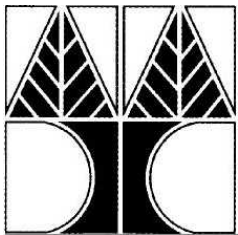
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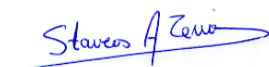
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Expert opinion elicitation in option pricing

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Abstract

In this paper we propose a model for option pricing where expert opinion is used to elicit estimates of unobserved volatilities of the underlying assets. We suggest two different pricing approaches based on the Black-Scholes model (Black and Scholes (1973)). The results of the paper shed light on the effect of uncertainty and choice of volatility probability distribution on European option prices. We examine the cases of financial and real option. For financial options we demonstrate that even under the assumption of constant volatility and investor homogeneity, as well as without the need to resort to more complex assumptions for the underlying's stochastic process, a smile effect is justified. For the more complex case of real options we demonstrate that by neglecting parameter uncertainty we can easily missprice options by 15-20%, and that misspricing is almost a linear function in time-to-maturity and the degree to which the option is in-the-money.

Key Words: Expert opinion; Arbitrage Pricing; Volatility; Options; Parameter estimation

1 Introduction

Nowadays, derivative contracts (e.g. options) form a very important part of the financial markets. One of the major problems in the valuation of derivative contracts is the accurate determination of a model for the movement of the prices of the underlying. Even in the case where we assume that the underlying follows a stochastic process in the form of a geometric Brownian motion there still remains the problem of the determination of the parameters involved in this simple, yet widely used in practice model. Of these parameters the most important is the volatility of the underlying.

The problem of determining the volatility of the underlying and the effects of its uncertainty on the prices of the derivatives has troubled the financial community since the introduction of the celebrated Black-Scholes model. A number of methods has been proposed (see e.g. Bunnin et al (2000), Darsinos and Satchell (2001) and Carolyi (1993) and the references therein). These methods are designed to extract information on the volatility of the underlying asset using the past information included in historical time series of the price of the underlying or the derivative asset.

The aim of this paper is to propose an alternative method for the determination of the probability distribution of the unobservable volatility for the

underlying, through the use of expert opinion. Then, we introduce a model for the pricing of the derivative asset which is a generalization of the usual Black-Scholes model (see Avellaneda and Laurence, (2000)). This enables us to assess the effects of uncertainty and the elicitation method on the derivative price. Our approach reproduces realistic effects such as the volatility smile observed in real financial markets. The proposed methodology emulates the actual way that financial decisions are taken, that is through the judgment of 'experts' (in our case investment managers, financial analysts, etc.) of a given situation. To our knowledge this is a novel approach in asset valuation, which can prove very helpful for certain classes of assets (e.g. non traded assets, real options etc) for which past data are usually unavailable. Moreover, this technique can be extended to a wide range of applications in financial decision making such as the assessment of management decisions, asset valuation and corporate finance.

The paper is organized as follows. Section 2 reviews basic elicitation techniques. In Section 3 we develop methodologies for the pricing of derivative assets on underlyings with uncertain parameters based on extensions of the Black-Scholes model. In Section 4 two specific approaches to model expert opinion are presented and their effects on the derivative prices are examined

in Section 5. Finally, concluding remarks are presented in Section 6.

2   General elicitation methods

Expert judgment has always been an important contributor in science and engineering. It is exceedingly regarded as an alternative type of data leading to the development of relevant quantitative techniques. There have been many ways to formulate the elicitation of the expert opinion.

More specifically, we are mostly interested in situations where the variable of interest assumes values in a continuous set. Typically, the expert is asked to provide information about the subjective probability distribution that best describes the unknown variable. This information can be formulated in many ways, such as the cumulative distribution function or certain parameters like the mean, the standard deviation or the quantiles. Therefore, Bayesian statistical inference, with its subjective nature, welcomes the use and application of opinion elicitation. In principle, a Bayesian analyst considers anyone's prior assumption-distribution as valid and worth exploring. However, the opinion conveyed by an expert can be of serious assistance. Important work in the field of expert performance, uncertainty analysis and

information measures has been established by Cooke (see for example 1991, 1994).

In this paper, we extend the results of Lindley and Singpurwalla (1996) and O'Hagan (1998) to option pricing and more specifically to the unknown volatility of the underlying asset(s). In the former work the unknown parameters are modeled using a lognormal probability distribution accounting for any common knowledge of the experts. In the latter case a probability distribution was specified grouping the results of several experts by fitting a beta distribution in the resulting histogram. The theoretical treatment in both cases is from a Bayesian point of view.

3 Black-Scholes models in the case of unknown parameters

In this section we propose two models for the valuation of European options when there are unknown parameters related to the underlying. We employ expert opinion for parameter estimation in the spirit of the previous section (for more details on methods and techniques utilized see Section 4). We assume that investors are homogeneous, and agree on the experts to be elicited.

Throughout our analysis, we assume the validity of the Black-Scholes frame-work with constant riskless rate r.

3.1 Financial assets

Assume a typical Black-Scholes market consisting of a riskless bond and a stock (Black and Scholes (1973)). The bond price follows the equation

$$dB_t = rB_t dt \tag{1}$$

while the stock price follows the stochastic differential equation

$$dS_t = \mu S_t dt + \tilde{\sigma}(\omega) S_t dW_t \tag{2}$$

where $\tilde{\sigma}(\omega)$ is an unobservable constant volatility which is estimated with some error. Expert opinion will be elicited for the random variable $\sigma(\omega)$.

Let us assume that for each realization of the random variable $\sigma(\omega)$ there are no arbitrage opportunities in the market, i.e. that for every ω there exists $\lambda(\omega)$ such that

$$\mu(\omega) - r = \tilde{\sigma}(\omega)\lambda(\omega) \tag{3}$$

where r is the risk free return rate. Using the no arbitrage assumption the local expected return μ_P and local volatility σ_P of any derivative security

should satisfy the relation

$$\mu_P - r = \tilde{\sigma}_P \lambda \tag{4}$$

whre λ is the Sharpe ratio of the derivative security.

Using Itô's lemma, we express μ_P and σ_P in terms of the price of the price $P = P(S_t, t)$ of the derivative security as follows:

$$
\begin{aligned}
\tilde{\sigma}_P &= \frac{1}{P}\frac{\partial P}{\partial S}S\tilde{\sigma}(\omega) &(5)\\
\mu_P &= \frac{1}{P}\left(\frac{\partial P}{\partial t} + \mu S\frac{\partial P}{\partial S} + \frac{1}{2}\tilde{\sigma}^2(\omega)S^2\frac{\partial^2 P}{\partial S^2}\right) &(6)
\end{aligned}
$$

We now proceed in two different ways to the valuation of the derivative product.

1. We assume that the Sharpe ratio for the derivative product is equal to the Sharpe ratio for the underlying for all ω. This leads to an equation of the form

$$\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\tilde{\sigma}^2(\omega)S^2\frac{\partial^2 P}{\partial S^2} - rP = 0 \tag{7}$$

which is the Black-Scholes partial differential equation for the value of a European type derivative product with unknown parameters. A reasonable choice for the price of the product would be the expectation of $P(S, t, \omega)$ over the distribution of the unknown parameters $\sigma(\omega)$, i.e.

$$P_1(S, t) = E[P(S, t, \omega)]. \tag{8}$$

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2. Alternatively, assume that the Sharpe ratios are equal on average.
This simplifying assumption (which is actually the standard in real practice)
leads to the following partial differential equation for the option price

$$\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}E[\tilde{\sigma}^2(\omega)]S^2\frac{\partial^2 P}{\partial S^2} - rP = 0 \qquad (9)$$

This yields an alternative valuation of the option denoted by $P_2(S,t)$.

We note that in general $P_1 \neq P_2$. Both versions of the Black-Scholes
equation derived here may be solved analytically using a reduction of the
equation to the heat equation. In the special case of the European call for
example, which at the maturity date has $P(S,T) = max(S - K, 0)$ we have
that

$$P_1 = E[S\,N(d_1(\tilde{\sigma}^2(\omega)))] - E[K exp(-r(T-t))N(d_2(\tilde{\sigma}^2(\omega)))] \quad (10)$$

$$P_2 = S\,N(d_1(E[\tilde{\sigma}^2(\omega)])) - K exp(-r(T-t))N(d_2(E[\tilde{\sigma}^2(\omega)])) \quad (11)$$

where

$$d_1(x) = \frac{1}{\sqrt{x(T-t)}} ln\left(\frac{S exp(r(T-t))}{K}\right) + \frac{\sqrt{x(T-t)}}{2}$$

$$d_2(x) = d_1(x) - \sqrt{x(T-t)}$$

$$N(x) = \frac{1}{2\pi}\int_{-\infty}^{x} exp\left(-\frac{y^2}{2}\right)dy$$

We will solve for P_1 using simulation (similarly with Hull and White
(1987)).

3.2 The case of non-traded assets or real assets

The above derivation of the Black-Scholes model for the case of the unknown parameters may be extended to cover the case of non-traded assets. This extension is interesting since it allows the methodology to cover the important case of real options (see for example Dixit and Pyndick (1994), McDonald and Siegel (1986) and Trigeorgis (1996)).

The extension to the case of non-traded assets can be achieved through the linkage of the non-traded asset to a traded asset with the use of a Capital Asset Pricing Model (CAPM). We make the usual assumption that the two assets are perfectly correlated.

The real asset is assumed to follow in the risk neutral measure the following stochastic differential equation

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t \tag{12}$$

where following McDonald and Siegel (1984), the dividend yield δ is defined as $\delta = R_s - g$. We now assume the validity of a continuous time CAPM as in Merton (1973), according to which

$$R_s = r + \beta_s(R_M - r)$$

where $\beta_S = \frac{\sigma_{s,M}}{\sigma_M^2} = \frac{\rho_{s,M}\sigma_s\sigma_M}{\sigma_M^2}$. In the above expressions r is the instantaneous

riskless rate of interest, R_M is the expected market return and the asset's beta coefficient β_S is a function of the market's variance σ_M^2 and the covariance with the market return $\sigma_{S,M}$. By $\rho_{S,M}$ we denote the correlation coefficient between the market and the asset.

Assuming spanning holds and using Itô's lemma we find for the option price $P(S, t)$ that

$$
\begin{aligned}
\frac{dP}{P} &= \frac{1}{P}\left(\frac{\partial P}{\partial t} + (r - \delta)S\frac{\partial P}{\partial S} + \frac{1}{2}\tilde{\sigma}^2 S^2\frac{\partial^2 P}{\partial S^2}\right)dt + \frac{S}{V}\frac{\partial^2 V}{\partial S}\sigma dW_t \\
&:= a_P dt + \tilde{\sigma}_P dW_t
\end{aligned}
\tag{13}
$$

In the risk neutral measure the return of the asset will have to be equal to the instantaneous riskless rate of interest r. This leads to the determination of the following Black-Scholes equation for the real asset

$$
\frac{\partial P}{\partial t} + (r - \delta)S\frac{\partial P}{\partial S} + \frac{1}{2}\tilde{\sigma}^2 S^2\frac{\partial^2 P}{\partial S^2} - rV = 0
\tag{14}
$$

where $r - \delta = g - \beta_S(R_M - r) = g - \frac{\rho_{S,M}\tilde{\sigma}\sigma_M}{\sigma_M^2}(R_M - r)$.

The option price will be $P_1 = E[P(S, t)]$. For the determination of this price we need to elicit the parameter $\tilde{\sigma}$. Alternatively, we may consider the price P_2 which is the solution of the above Black-Scholes equation with coefficients the averages of the random coefficients of this equation. This seems to be the standard assumption in practice.

The Black-Scholes model (as for example in McDonald and Siegel (1985))
for the real asset with unknown parameters, may be used to assess the impact
of volatility on real option pricing.

4   Elicitation of the expert opinion

The parameter of interest is the variance of the volatility of one underlying
asset. This is reflected through the standard deviation $\tilde{\sigma}^2(\omega)$. The following
approaches are suggested.

4.1   Using a lognormal distribution

Consider k experts, offering their opinions about the unknown parameter
$\theta = \log \tilde{\sigma}^2$, i.e., the logarithm of the random variance of the underlying asset.
We assume that $\tilde{\sigma}^2$ follows a lognormal probability distribution. Thus, θ is
distributed following a normal distribution with m_i and s_i denoting the mean
and standard deviation respectively elicited by the i-th expert $(i = 1, \ldots, k)$.
The decision maker opts to write $m_i = \beta_i \theta + \alpha_i$ where α_i and β_i adjust
the subjective mean to avoid "location biases" and multiplies s_i by τ_i where
$\tau_i \neq 1$ to inflate or deflate the expert's estimate of the uncertainty to avoid

a "scale bias".

We denote by $p(\theta|m, s)$ the probability distribution of the unknown parameter θ given $m = (m_1, \ldots, m_k)$ and $s = (s_1, \ldots, s_k)$ which is proportional to $p(m, s|\theta)p(\theta)$ or equivalently to $p(m|s, \theta)p(s|\theta)p(\theta)$.

Following Lindley (1983), we assume that $p(s|\theta)$ does not depend on θ, therefore providing no knowledge of the parameter, $p(\theta)$ is constant and $p(m|s, \theta)$ is multivariate normal with m_i having mean $m_i = \theta\beta_i + \alpha_i$, standard deviation $\sigma_i s_i$ and correlation ρ_{ij} between m_i and m_j. Based on this, our final opinion about θ is that it is normally distributed with mean

$$\mu = \frac{\sum_{i,j} \beta_i \sigma^{ij}(m_i - \alpha_j)}{\sum_{i,j} \beta_i \sigma^{ij}\beta_j} \tag{15}$$

and standard deviation

$$\sigma = [\sum_{i,j} \beta_i \sigma^{ij}\beta_j]^{-0.5} \tag{16}$$

where σ^{ij} are the elements of the inverse matrix with elements $\sigma_{ij} = \rho(\sigma_{ii}\sigma_{jj})^{0.5}$ and $\sigma_{ii} = \tau_i^2 s_i^2$.

4.2 Fitting a Beta distribution

Initially, the experts agree on an upper and on a lower bound, U and L respectively, such that the parameter will almost certainly lie between them.

These bounds have to be widened, if even one expert argues for a wider support of the distribution.

The next step for the experts is to agree on the mode M. The first stage of the expert elicitation, concludes with the elicitation of the following probabilities.

$$p_1 = P(L \leq \tilde{\sigma}^2 \leq \frac{(L+M)}{2}) \tag{17}$$

$$p_2 = P(\frac{(L+M)}{2} \leq \tilde{\sigma}^2 \leq \frac{(L+3M)}{4}) \tag{18}$$

$$p_3 = P(\frac{(L+3M)}{4} \leq \tilde{\sigma}^2 \leq M) \tag{19}$$

$$p_4 = P(M \leq \tilde{\sigma}^2 \leq \frac{(3M+U)}{4}) \tag{20}$$

$$p_5 = P(\frac{(3M+U)}{4} \leq \tilde{\sigma}^2 \leq \frac{(U+M)}{2}) \tag{21}$$

$$p_6 = P(\frac{(M+U)}{2} \leq \tilde{\sigma}^2 \leq U) \tag{22}$$

The use of the mode enables us to "capture" the majority of the probabilities and discourages the elicitation of small probabilities. A histogram was formed based on the experts' assessments of $q_1, \ldots, q_6$ and a beta distribution having support (L,U) was fitted to the histogram.

If the experts did not agree with the final outcome of the distribution, a "stepping-back" operation was suggested, adjusting L and U based on the

following formula:

$$L_{NEW} = \min\{\frac{L+M}{2}, M - F(M-L)\} \tag{23}$$

$$U_{NEW} = \max\{\frac{M+U}{2}, M + F(U-M)\} \tag{24}$$

$$F = \frac{3Var(\tilde{\sigma}^2)}{\min(M-L, U-M)} \tag{25}$$

The rationale for this algorithm is that it shifts L and U a distance of $3Var(\tilde{\sigma}^2)$ from the mode while keeping the ratio $\frac{M-L}{U-M}$ fixed to avoid skewness. Based on L_{NEW} and U_{NEW}, a new Beta distribution is fitted and the experts decide if it reflects their opinions better.

5 The effect of volatility elicitation on option prices

The effect of the unknown parameters on the Black-Scholes valuation may be assessed by evaluation of the proposed Black-Scholes price P_1. The expectation in this formula can be calculated under the probability distribution of the volatility parameters provided by the set of experts. In the case where the methodology of Singpurwalla and Lindley is used, this averaging will be performed over a lognormal distribution with mean and standard deviation

provided by equations (15) and (16) where if the methodology of O'Hagan is used this averaging will be performed over a Beta distribution with parameters provided by the fitting procedure.

In an attempt to obtain a first assessment of the effect of expert opinion and parameter uncertainty in option pricing we have calculated the price $P_1 = E[P]$ in the cases where $\sigma(\omega)$ is distributed by a lognormal or a Beta distribution. We assume that the set of experts is similar in both cases, in the sense that the two distributions have equal means and standard deviations. The differences in prices of the one asset call option are shown in Figures 1 and 2 for different values of the time to expiry. The interest rate is 10%, the strike price is 95 and the mean volatility squared is 0.5 and the standard deviation is 0.01. The dotted line is P_1(lognormal)-P_1(Beta), the starred line is P_1(lognormal)-P_2 and the full line is P_1(Beta)-P_2. We observe that depending on the value of the time to expiry, the different types of valuation may change from underpricing to overpricing of the option. This result is expected by the derivation of the Vega of the option and its convexity properties. We also observe that the differences in the various prices depend on the standard deviation of the volatility elicited by each group of experts. The pattern arising from our observations is the following: for small values of the

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standard deviation of the volatility the use of Beta distribution and lognormal distribution provide almost identical values for the option price except near the strike value where the Beta distribution provides lower estimates of the price of the option (see Figure 1). The situation reverses for larger values of the standard deviation of the volatility where the two prices for lognormal and Beta agree only around the strike price. In the other ranges of values, the Beta distribution gives higher option price estimates (see Figure 2).

Another point of interest that is addressed through the numerical work is the effect of the composition of the sample of experts on the option price. This may be studied by assessing the effect of the standard deviation of the volatility parameter to be elicited, on the option price. We expect that a group of experts which is almost homogeneous in its composure will provide a volatility which will be a random variable with small value of the standard deviation while a heterogeneous group of experts will provide a volatility which will be a random variable with large standard deviation. The effect of the standard deviation of the volatility is shown in Figures 3-4 for the lognormal and the Beta distributions respectively. It is observed that the Beta distribution provides sharper differences around the strike value.

These results may be explained by examining the variation of the call price

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with S and the volatility (see Figure 5) and the variation of the sensitivity of the call price with respect to the volatility, that is the option's Vega= $\frac{\partial C}{\partial \sigma}$, as a function of S and the volatility (see Figure 6). It is observed that for small values of the volatility, the sensitivity of the option price on volatility is significant only around the strike price K and the option price is insensitive to volatility elsewhere. This situation changes for large values of the volatility where the option price is insensitive on changes of the volatility only for small values of the price of the underlying S. As can be seen from the probability density functions of the lognormal and the beta distribution, for large values of the standard deviation of the volatility, the contribution of realizations of the sample of the Beta distribution having large values of σ is important. The contribution of these large volatility values accounts for the large sensitivity of the option price on volatility and hence provides an explanation of systematic overpricing or underpricing of the option when using the beta distribution with large standard deviation. On the other hand, when the standard deviation is small the sample consists of small values of volatility (near 0.5) and by observing the plot of Vega we conclude that the sensitivity of the option price on volatility is significant only near the strike price. Therefore, the option price is insensitive on the choice of volatility

distribution everywhere but near the strike price.

An important observation resulting from the analysis is that, under assumptions of constant volatility, the phenomenon known as volatility smile (see e.g. Derman et al (1994), Dupire (1994), Rubinstein (1994)) is justified, without the need to assume heterogeneous investors or a more complex stochastic process like jump-diffusion or stochastic volatility. Figures 7 and 8 demonstrate the implied volatility from the call option prices as a function of the underlying price for fixed strike price K, for expiry dates 0.25 and 1 year respectively. The figures reproduce the observed phenomenon of non-constant implied volatility (widely referred to as the smile effect). The slope of the smile curve is seen to depend on the choice of distribution function for the volatility. In particular if the volatility is distributed according to the beta distribution the smile is 'steeper' than the case of the lognormal distribution. This comes to no surprise since the beta distribution has heavier tails than the lognormal. Furthermore, we observe that regardless of the choice of distribution, the smile is more pronounced for larger values of the standard deviation. However, even a small parameter uncertainty about the Black and Scholes volatility can create a smile effect of a significant magnitude. The true volatility seems to be close to the one derived for slightly

out-of-the-money call contracts, not exactly at-the-money ones.

We next examine the effect of volatility on real option pricing. In Figure 9 we present the difference between the proposed prices of real options for different times to expiry. It can be seen that P_1 underprices the option for values of the asset above a critical value which is now below the strike price while the values P_1 and P_2 are almost identical for values of the asset below that critical price. In Figure 10 we present the same quantities as before but for larger value of the standard deviation of the volatility. It is seen that for larger values of the standard deviation P_1 may overprice the option for small enough values of the asset price. Finally, in Figures 11-12 we present the difference in P_1 using the lognormal and Beta distribution respectively for two different standard deviations of the volatility parameter (0.01 vs 0.1). All other parameters are the same as in Figure 1. The panels correspond to different times to expiry as in Figure 9. As before, we observe that the most sensitive region is near the strike price but clearly below it, a fact that is conjectured to be due to effect of volatility on the dividend yield which is present in the real options case (see Sarkar (2000)). It is also worth observing that by neglecting parameter uncertainty we can easily missprice options by 15-20function in time-to-maturity and the degree to which the option is in-

the-money.

6 Conclusions

In summary, this paper proposed a novel approach through the use of expert opinion to the pricing of derivative assets when the parameters of the underlying are subject to uncertainty. We focus on the case of the unobservable volatility employing expert opinion. The lognormal and Beta distributions are utilized to provide estimates of the volatility and an extension of the Black-Scholes model for unknown parameters is used to determine the price of the financial and real options.

One of the main results is that for small values of the standard deviation of the volatility both distributions differ only near the strike value where the lognormal distribution yields higher estimates of the option price. On the contrary, when the standard deviation of the volatility increases, the only area where the two distributions demonstrate similar results is near the strike value. Furthermore, in this area the Beta distribution presents sharper differences than the lognormal.

Another important observation is that even under small uncertainty in

the Black-Scholes volatility, the implied volatility from the call prices demonstrates smile effects. It is interesting to see that we reproduce smile effects, within the framework of a Black-Scholes model, without the need to assume heterogeneous investors or a more complex stochastic process like jump-diffusion or stochastic volatility.

In the case of real options, an intertemporal Capital Asset Pricing Model is used to associate the return of the real asset with the market portfolio. Thus, the unobservable volatility introduces uncertainty to the dividend yield of the asset. Through the aforementioned elicitation procedure and the use of a modified Black-Scholes equation, the effect of uncertainty in the price of real options is assessed. It is seen that neglecting uncertainty in the parameters may result in important mispricing of the option.

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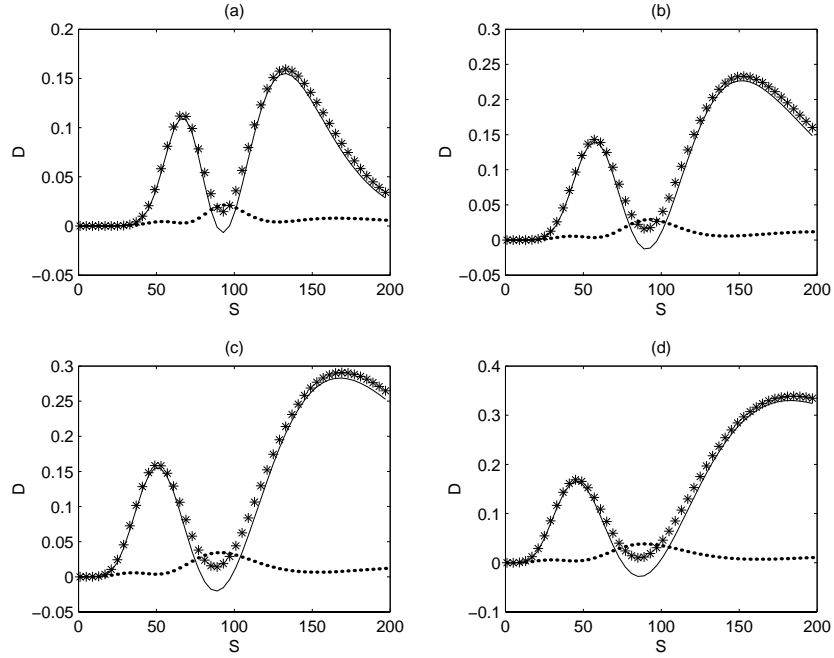
Figure 1: Difference between the proposed prices for different times to expiry. The interest rate is 10%, the strike price is 95 and the mean volatility squared is 0.5 and the standard deviation is 0.01. The dotted line is P_1(lognormal)-P_1(Beta), the starred line is P_1(lognormal)-P_2 and the full line is P_1(Beta)-P_2. Plot (a) corresponds to $T-t = 0.25$ years, plot (b) corresponds to $T-t = 0.5$ years, plot (c) corresponds to $T-t = 0.75$ years and plot (d) corresponds to $T-t = 1$ year.

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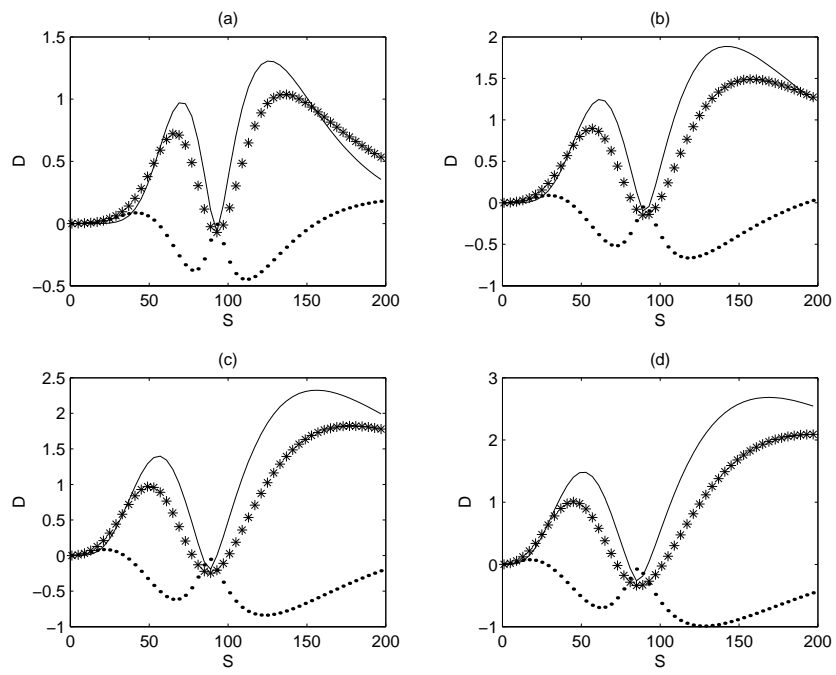
Figure 2: Same as Figure 1 but with standard deviation of the volatility 0.1
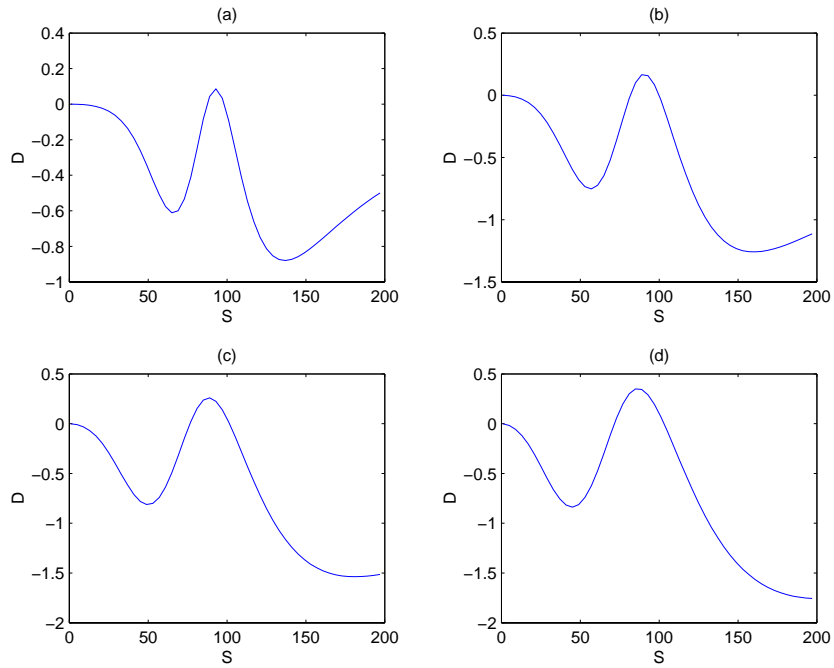
Figure 3: Difference between P_1 (lognormal) for two different standard deviations of the volatility parameter (P_1 (std 0.01) - P_1 (std 0.1). All other parameters are the same as in Figure 1. The plots correspond to different times to expiry as in Figure 1
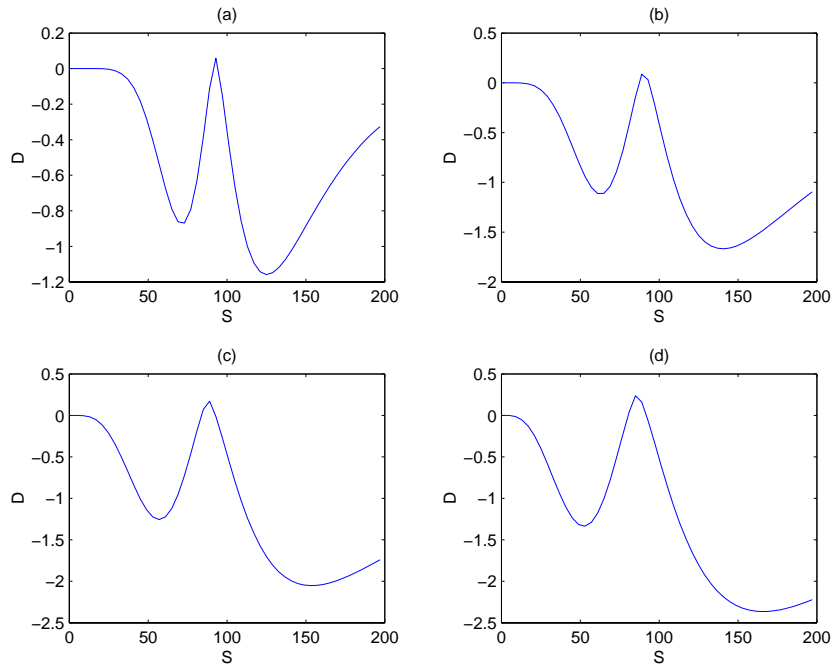
Figure 4: Difference between P_1 (Beta) for two different standard deviations of the volatility parameter(P_1 (std 0.01) - P_1 (std 0.1). All other parameters are the same as in Figure 1. The plots correspond to different times to expiry as in Figure 1
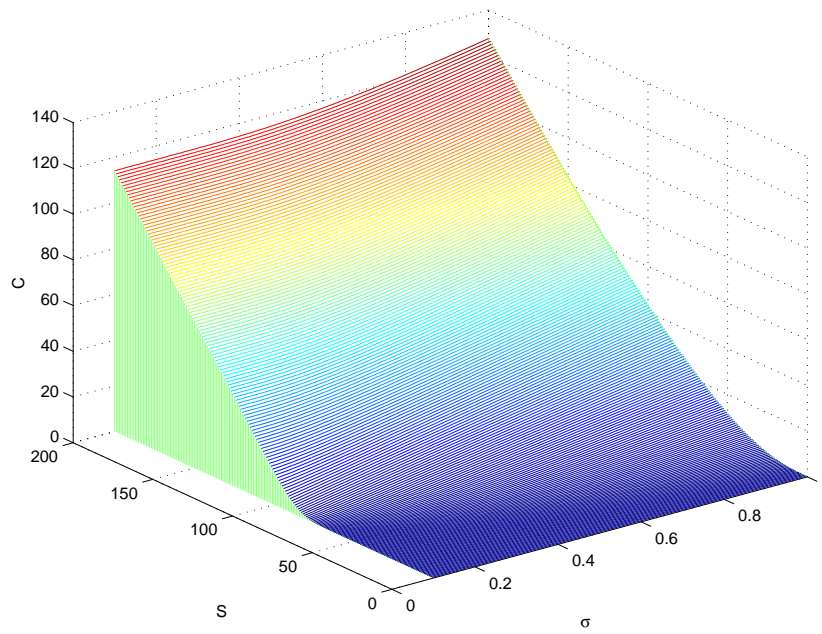
Figure 5: The call price C as a function of the price of the underlying S and the volatility σ. The time to expiry is 1 year, the strike price is $K = 95$ and the interest rate is 0.1.
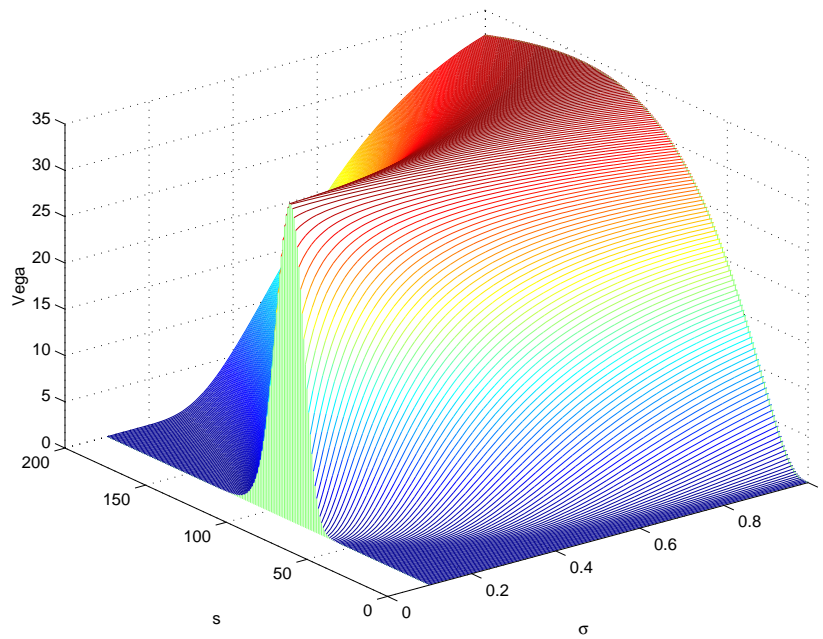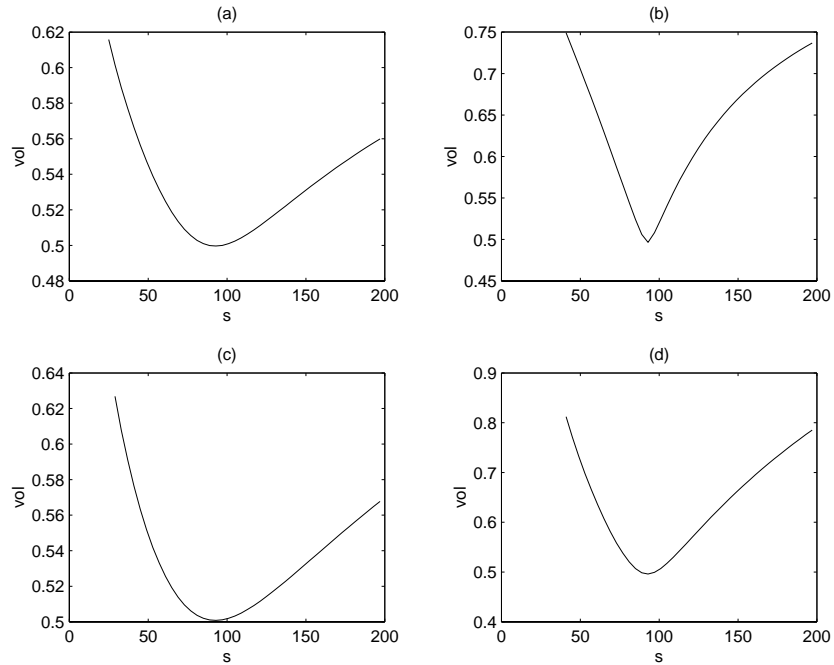
Figure 6: The Vega of the call ($Vega = \frac{\partial C}{\partial \sigma}$) as a function of the price of the underlying S and the volatility σ. The time to expiry is 1 year, the strike price is $K = 95$ and the interest rate is 0.1.

Figure 7: The implied volatility as a function of S for constant K. Plot (a) corresponds to a Beta distribution of uncertain volatility with standard deviation 0.01, plot (b) corresponds to a Beta distribution of uncertain volatility with standard deviation 0.1, plot (c) corresponds to a lognormal distribution of uncertain volatility with standard deviation 0.01 and plot (d) corresponds to a lognormal distribution of uncertain volatility with standard deviation 0.1. In all cases the time to expiry is 0.25 years, strike price is 95 and the mean volatility is 0.5.
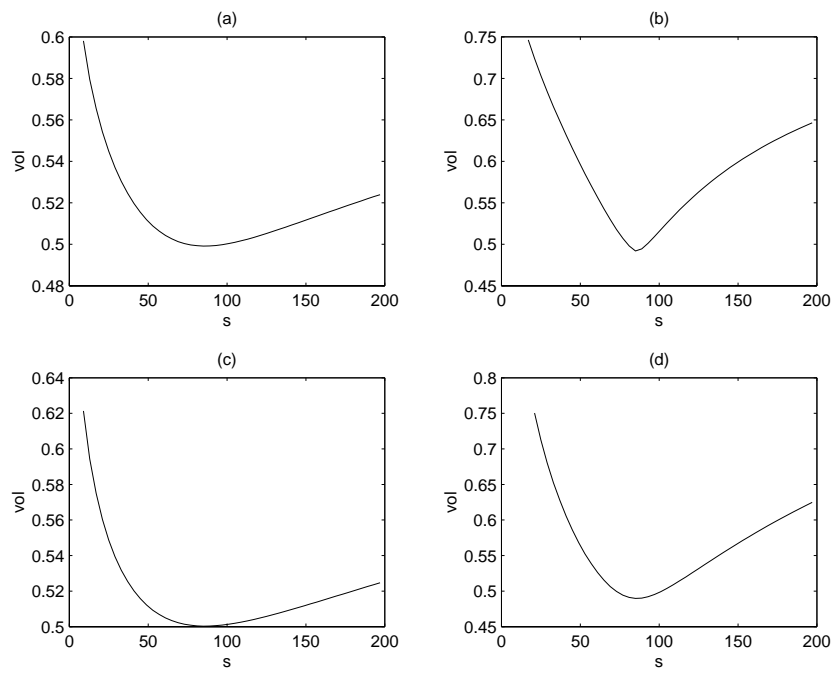
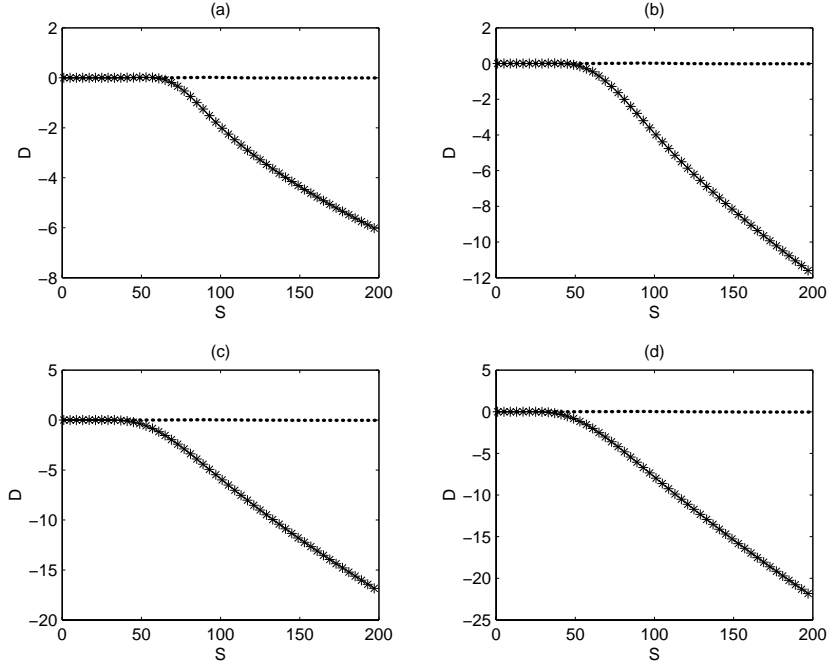Figure 8: As in Figure 7 but the time to expiry is 1 year.

Figure 9: Difference between the proposed prices of real options for different times to expiry. The interest rate is 5%, the strike price is 95 and the mean volatility squared is 0.5 and the standard deviation is 0.01. The correlation with the market portfolio is $\rho_{S,M} = 0.5$, $\sigma_M = 0.1$, $g = 5\%$ and $R_M = 10\%$. The dotted line is P_1(lognormal)-P_1(Beta), the starred line is P_1(lognormal)-P_2 and the full line is P_1(Beta)-P_2. Plot (a) corresponds to $T - t = 0.25$ years, plot (b) corresponds to $T - t = 0.5$ years, plot (c) corresponds to $T - t = 0.75$ years and plot (d) corresponds to $T - t = 1$ year.
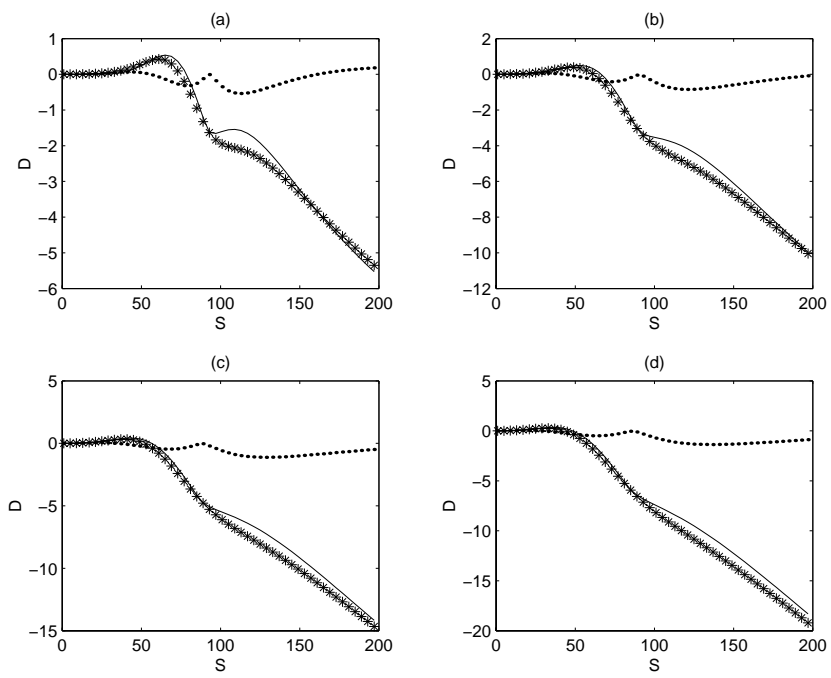
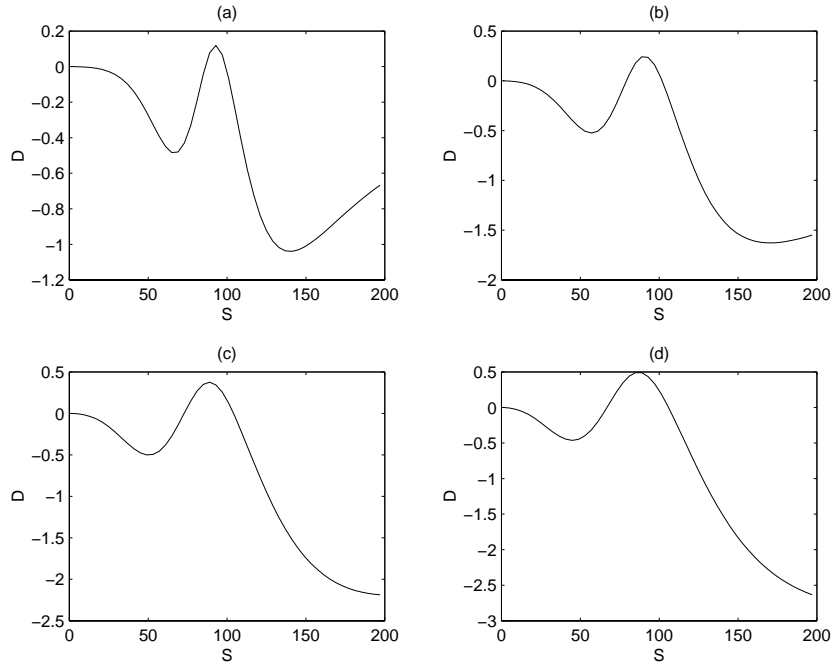Figure 10: Same as Figure 9 but with standard deviation of the volatility 0.1

Figure 11: Difference between P_1 (lognormal) for two different standard deviations of the volatility parameter (P_1 (std 0.01) - P_1 (std 0.1). All other parameters are the same as in Figure 1. The plots correspond to different times to expiry as in Figure 9
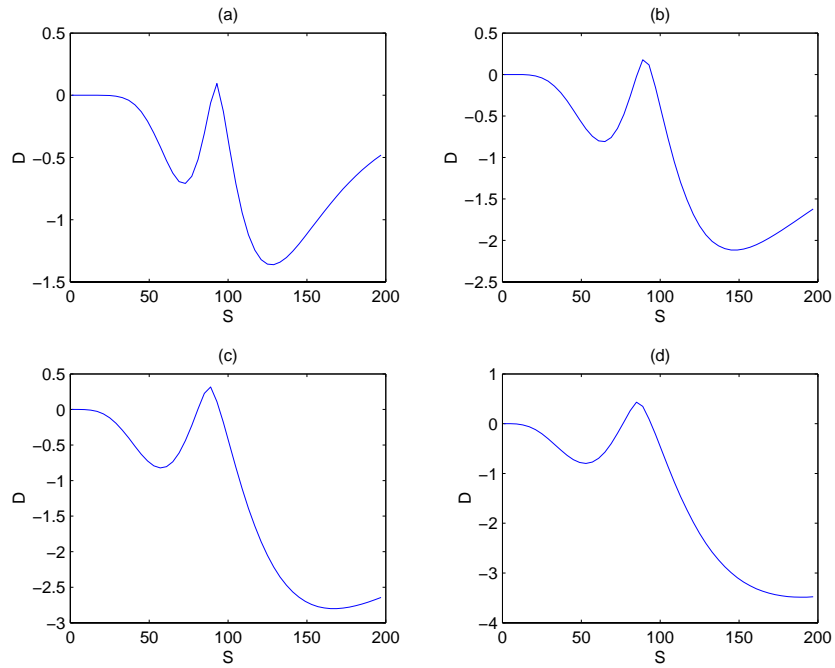
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Figure 12: Difference between P_1 (Beta) for two different standard deviations of the volatility parameter(P_1 (std 0.01) - P_1 (std 0.1). All other parameters are the same as in Figure 1. The plots correspond to different times to expiry as in Figure 9