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Hysteresis Models of Investment with Multiple Uncertainties and Exchange Rate Risk

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Abstract. In this article we use the hysteresis model of investment developed by Brennan and Schwartz, and Dixit, and we extend it to capture the impact of interacting uncertainties on a firm with foreign operations. We develop a three-country, four-factor model where both continuous revenues and continuous costs are stochastic and are generated in countries other than the home country of the investor, who has to carry foreign currencies' risk. All four state-variables follow geometric Brownian motion processes. A critical assumption is made that the capital outlays for switching between the idle and the active states are constant fractions of the costs. An efficient numerical solution is used to demonstrate applications of the model on a multinational corporation facing operating and exchange rate risks in a multistage investment setting with interacting investment and operating options.

Key words: hysteresis, multiplicative uncertainties, exchange rate risk, compound (real) option, flexibility

JEL Classification: D92, F23, G13

Introduction

Partial irreversibility was first introduced in the literature by Brennan and Schwartz (1985); soon thereafter Dixit (1989a, 1989b, 1989c) generalized their results and coined the term *hysteresis* for the zone of inaction. Specifically, this zone is the area between the upper and the lower critical asset price. The upper critical price defines when we should invest, thereby incurring capital cost, receiving variable revenues, and paying variable costs. The lower critical price defines when we should disinvest, similarly incurring (closing) capital costs. In the zone in between, if we have not invested, we remain so; likewise, if we have invested, we remain invested. Thus, crossing either the upper or the lower boundary determines an action (to invest or to disinvest), that depends on the previous state. This path dependency makes the valuation of such investment options difficult. Under the assumption of infinitely lived opportunities we arrive at the solution of a system of four highly nonlinear equations.

Brennan and Schwartz (1985) valued natural resource investments and demonstrated that the classic Net Present Value (NPV) rule fails under uncertainty and irreversibility-inducing sunk costs. Dixit (1989a) used the model to show the hysteresis effects in a simplified two-sector economy with costly capital mobility, and also demonstrated that the hysteresis band can arise even under certainty. Uncertainty widens this band. Dixit (1989b) used the model

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to show the hysteresis effect in entering in, or exiting from, a foreign market when the exchange rate follows geometric Brownian motion. This model considers the more general case of industry equilibrium when the company has the (sequential) option to purchase or abandon many existing producers. Extensions of the hysteresis model have also appeared in corporate finance (Mauer and Triantis, 1994), and under foreign exchange as the sole factor of risk (Mello, Parsons, and Triantis, 1995; Bell, 1995).

In this paper we aim to extend the hysteresis model of investment to capture the effects of multiple uncertainties. In general, we consider both the underlying continuous revenues and costs to be stochastic; furthermore, both are multiplicative functions with stochastic exchange rates. All four state-variables follow geometric Brownian motion processes. A first effort to include multiplicative uncertainties appeared in the case studies of Crousillat and Martzoukos (1991), in the context of complete irreversibility and without hysteresis. There, both the underlying variable and the exercise price are multiplicative functions of two uncertainties, but the correlation between the two underlying assets, and their drift terms, are derived subjectively.

We proceed as follows. In the first section we review the hysteresis model of investment. In the second section we extend the hysteresis model to the case of stochastic variable costs and exchange rate risk affecting both the continuous revenues and the costs. The third section provides a set of applications, and the last section summarizes and concludes. In the Appendix we demonstrate a numerical solution to the general perpetual horizon hysteresis problem. This method's efficiency is instrumental for the applications we provide.

1. Review of the hysteresis model of investment

In the standard model, valuation of a claim (or *real option*) V is contingent on the stochastic and continuous cash flow R, which follows a geometric Brownian motion process with drift m and instantaneous variance σ^2. The methods of valuation have been established in the literature that applies stochastic calculus to valuation of options. Standard results from stochastic calculus, and application in finance and option valuation can be found in Malliaris and Brock (1982), Karatzas and Shreve (1991), Black and Scholes (1973), and Merton (1973a, 1973b) among others. Review of the literature on contingent claims valuation of investments can be found in Pindyck (1991), Dixit (1992), Dixit and Pindyck (1994), and Trigeorgis (1996). In general, a continuous time capital asset pricing model (see Merton, 1973b, or Breeden, 1979) is assumed to hold.

Generally, a continuous cost W would be incurred to sustain the revenue R. The difference between the discount rate and the growth rate of the revenues is denoted by δ, and is an opportunity cost of deferring investment in the revenue producing project (see McDonald and Siegel, 1984); for a convenience yield interpretation, see Brennan and Schwartz (1985), and Brennan (1991). In general, we assume the absence of market imperfections (taxes, etc.) and an all-equity firm. In the following we stay closer to the more recent Dixit (1989a) paper.

The differential equation for the value of the project is

$$0.5\sigma^2 R^2 V_{RR} + (r - \delta) R V_R - rV + R - W = 0,$$

where V_R, V_{RR} denote derivatives, and r refers to the exogenously determined riskless rate of interest. The absence of a time derivative implies an infinite horizon for both the investment opportunity and the underlying cash flows. In the above differential equation the term $R - W$ is present when the project is active. When it is idle, the investment option follows a similar differential equation

$$0.5\sigma^2 R^2 V_{RR} + (r - \delta)RV_R - rV = 0.$$

The value of an active project takes the form

$$V_1 = R/\delta - W/r + AR^{-a}, \tag{1}$$

where the first two terms equal the present value of the expected cash flows, and the last term equals the value of the option to temporarily abandon. For an inactive project, the value of the investment option equals

$$V_0 = BR^{\beta}. \tag{2}$$

The terms $-a$ and β are the two roots of the quadratic equation

$$f(Q) = 0.5\sigma^2 Q(Q - 1) + (r - \delta)Q - r = 0,$$

and they equal

$$0.5 - (r - \delta)/\sigma^2 \pm \surd\{[(r - \delta)/\sigma^2 - 0.5]^2 + 2r/\sigma^2\}. \tag{3}$$

Root β with the positive second part is greater than unity, and root $-a$ is negative.

The two valuation equations involve the unknown values of the active project V_1, the idle V_0 project, and the parameters A and B. We need two more equations to solve this infinite horizon problem. Since these equations will involve two new unknowns for the trigger points (the upper and lower critical levels of revenue) we actually need four new equations. The first two are the value-matching conditions

$$\begin{aligned} V_1(R_H) - k &= V_0(R_H), \\ V_0(R_L) - l &= V_1(R_L), \end{aligned} \tag{4}$$

and two more are provided by the smooth pasting conditions

$$\begin{aligned} V_{1R}(R_H) &= V_{0R}(R_H), \\ V_{0R}(R_L) &= V_{1R}(R_L). \end{aligned} \tag{5}$$

In the above, R_H and R_L denote the critical values of the cash flow at which investment or disinvestment is triggered; subscript R implies the derivative. The sunk capital costs k and l are the costs needed to either invest or disinvest. The value functions impose continuity on the value of the investment before and after the trigger to either invest or disinvest. This is equivalent to precluding arbitrage at these trigger points. The smooth pasting conditions imply continuity of the derivatives.

The values V_1 and V_0 are substituted into these four conditions, and a nonlinear system of four equations with four unknowns (the parameters A and B, and the trigger points R_H and R_L) must be solved. They can be solved iteratively through a four-dimensional Newton-Raphson method. An approximate solution to this problem is given by Dixit (1991), who has also studied the convergence properties of this nonlinear system. The method proposed in the Appendix reduces the numerical solution to the most appropriate two-dimensional Newton–Raphson scheme and is very efficient.

For the purposes of this paper and for reasons that will become apparent in the next section, we must demonstrate that the homogeneity property of the option values in the underlying asset and the exercise price (shown in Merton, 1973) also applies to the hysteresis model. The question involves the terms AR^{-a} and BR^{β} that are included in the valuation equations (1) and (2), respectively, for the active V_1 and the idle V_0 project. We must show that in these two terms we can replace R with the ratio R/W, and then multiply the two terms by W without affecting the results (see Appendix for the numerical solution methodology). The parameters A and B are both functions of R_H and R_L (again, refer to Appendix), equations (A2) for A and (A3) for B. After replacing R_H and R_L with R_H/W and R_L/W, the parameters A and B become functions of $W^{-1-\alpha}$ and $W^{-1+\beta}$. But the terms $W(R/W)^{-a}$ and $W(R/W)^{\beta}$ are functions of $W^{\alpha+1}$ and $W^{-\beta+1}$ and the term W cancels out. Note that in the value-matching conditions (4), the switching costs k and l must also be replaced with k/W and l/W, an important point with implications that will become clear in the next section. Thus, we have demonstrated that the hysteresis option is homogeneous of degree 1 in (R, W, k, l).

2. Stochastic costs and exchange rate risk

To price a claim dependent on stochastic state-variables, we draw on Constantinides (1978), Harrison and Kreps (1979), Harrison and Pliska (1981), and Cox, Ingersoll, and Ross (1985). We assume that both the revenues R and costs W are stochastic, drawing on McDonald and Siegel (1986), who consider the real option to wait-to-invest with a stochastic exercise price (in the context of complete irreversibility and the absence of hysteresis). Furthermore, both cash flows are products of two uncertainties, and all four state-variables follow geometric Brownian motion processes. R is the product of AR and the exchange rate ER, and W is the product of AW and the exchange rate EW. The fixed costs to invest $k = k_q W$, and to disinvest $l = l_q W$ are constant fractions of W. We price the contingent claim under risk-neutrality from the perspective of the option holder. The foreign currencies follow (see Garman and Kohlhagen, 1983; Grabbe, 1983) the stochastic processes

$$dER/ER = (r - r_R)\,dt + \sigma_{ER}\,dz_{ER},$$

and

$$dEW/EW = (r - r_W)\,dt + \sigma_{EW}\,dz_{EW},$$

where r is the local, and r_R, r_W are the foreign riskless rates of interest. For all three countries we assume that the real and nominal rates are the same (no inflation). The two foreign assets before they are translated to the option holder's currency (the original reference from an options pricing perspective is Reiner, 1992, and Kat and Roozen, 1994, provide explicitly the risk-neutral process), follow

$$dAR/AR = (r_R - \delta_{AR} - \sigma_{AR,ER})\,dt + \sigma_{AR}\,dz_{AR},$$

and

$$dAW/AW = (r_W - \delta_{AW} - \sigma_{AW,EW})\,dt + \sigma_{AW}\,dz_{AW}.$$

We see that the risk-neutral drifts include not only the (local) dividend yields, but also the instantaneous covariance between the exchange rate and the cash flow (see *Siegel's paradox* in Hull, 1997, pp. 298–301). It is known that the partial differential equation (PDE) for the claim dependent on AR, ER, AW, and EW is

$$\begin{aligned}&0.5\sigma_{AR}^2 AR^2\, V_{AR,AR} + 0.5\sigma_{AW}^2 AW^2\, V_{AW,AW} + 0.5\sigma_{ER}^2 ER^2\, V_{ER,ER}\\ &\quad + 0.5\sigma_{EW}^2 EW^2\, V_{EW,EW} + \sigma_{AR,ER} AR\, ER\, V_{AR,ER} + \sigma_{AW,EW} AW\, EW\, V_{AW,EW}\\ &\quad + \sigma_{AR,EW} AR\, EW\, V_{AR,EW} + \sigma_{ER,AW} ER\, AW\, V_{ER,AW} + \sigma_{AR,AW} AR\, AW\, V_{AR,AW}\\ &\quad + \sigma_{ER,EW} ER\, EW\, V_{ER,EW} + (r_R - \delta_{AR} - \sigma_{AR,ER}) AR\, V_{AR} + (r_W - \delta_{AW}\\ &\quad - \sigma_{AW,EW}) AW\, V_{AW} + (r - r_R) ER\, V_{ER} + (r - r_W) EW\, V_{EW} - rV + AR\, ER\\ &\quad - AW\, EW = 0.\end{aligned}$$

The solution to such a problem would be practically infeasible if it were not for the specific structure that we impose. We will be able to reduce the dimensionality of the problem by using the costs W as a numeraire, since we have demonstrated in the previous section that the homogeneity property of the option values applies to the hysteresis model. To simplify the exposition we will first use the standard Ito calculus tools and reduce the PDE from one of four state-variables to one of two asset prices, the cash flows R and W. Using the multi-dimensional form of Ito's lemma it can be easily shown that from the perspective of the option holder and under risk-neutrality

$$dR/R = (r - \delta_{AR})\,dt + \sigma_R\,dz_R,$$

and

$$dW/W = (r - \delta_{AW})\,dt + \sigma_W\,dz_W,$$

with

$$\begin{aligned}\sigma_R^2 &= \sigma_{AR}^2 + \sigma_{ER}^2 + 2\sigma_{AR,ER},\\ \sigma_W^2 &= \sigma_{AW}^2 + \sigma_{EW}^2 + 2\sigma_{AW,EW},\\ \sigma_{R,W} &= \sigma_{AR,AW} + \sigma_{AR,EW} + \sigma_{ER,AW} + \sigma_{ER,EW}.\end{aligned} \tag{6}$$

The two-dimensional PDE for the active project is

$$0.5\sigma_R^2 R^2 V_{RR} + (r - \delta_{AR}) R V_R + 0.5\sigma_W^2 W^2 V_{WW} + (r - \delta_{AW}) W V_W + \sigma_{R,W} R W V_{RW} - rV + R - W = 0,$$

and a similar PDE obtains for the idle project (without the term $R - W$). The two value-matching conditions are

$$\begin{aligned} V_1(R_H, W) - k_q W &= V_0(R_H, W) \\ V_0(R_L, W) - l_q W &= V_1(R_L, W), \end{aligned} \tag{7}$$

with four smooth-pasting conditions

$$\begin{aligned} V_{1R}(R_H, W) &= V_{0R}(R_H, W) \\ V_{0R}(R_L, W) &= V_{1R}(R_L, W) \\ V_{1W}(R_H, W) - k_q &= V_{0W}(R_H, W) \\ V_{0W}(R_L, W) - l_q &= V_{1W}(R_L, W). \end{aligned} \tag{8}$$

Using

$$V(R, W, k, l) = W v(R/W) = W v(R'),\ R' = R/W$$

and

$$\begin{aligned} V_R &= \cdots = v_{R'} \\ V_W &= \cdots = v - R' v_{R'} \\ V_{RR} &= \cdots = v_{R'R'}/W \\ V_{WW} &= \cdots = R'^2 v_{R'R'}/W \\ V_{RW} &= \cdots = -R' v_{R'R'}/W \end{aligned}$$

we substitute into the earlier PDE, to finally get an ordinary differential equation

$$0.5\sigma^2 R'^2 v_{RR} + (\delta_{AW} - \delta_{AR}) R' v_R - \delta_{AW} v = 0,$$

with the variance defined by

$$\sigma^2 = \sigma_R^2 + \sigma_W^2 - 2\sigma_{R,W}. \tag{9}$$

We also get the new value-matching conditions

$$\begin{aligned} v_1(R'_H) - k_q &= v_0(R'_H) \\ v_0(R'_L) - l_q &= v_1(R'_L) \end{aligned} \tag{10}$$

and the new smooth-pasting conditions

$$\begin{aligned}
&v_{1R'}(R'_H) = v_{0R'}(R'_H)\\
&v_{0R'}(R'_L) = v_{1R'}(R'_L)\\
&v_1(R'_H) - k_q - R'_H v_{1R'}(R'_H) = v_0(R'_H) - R'_H v_{0R'}(R'_H)\\
&v_0(R'_L) - l_q - R'_L v_{0R'}(R'_L) = v_1(R'_L) - R'_L v_{1R'}(R'_L)
\end{aligned} \tag{11}$$

where the last two smooth-pasting conditions can be derived from the first two and the value-matching conditions (10), and are thus redundant. As in Margrabe (1978), and since the homogeneity property holds, we have reduced the dimensionality of the problem *as-if* only one variable is stochastic. One critical assumption is emphasized: Due to the boundary (value-matching) conditions given in equations (4) and (10), the switching costs to invest k and to disinvest l must be constant fractions of W, and in effect, stochastic. This is the same assumption that was used in Carr (1988), and which allowed the extension of Geske's (1977) compound option to the case of stochastic exercise prices using Margrabe's (1978) results. In the resulting equations, we need to remultiply by W in order to rescale.

Finally, the value of the active project equals

$$V_1 = R/\delta_R - W/\delta_W + AR^{-a}, \tag{12}$$

and the value of the option to invest when the project is idle equals

$$V_0 = BR^{\beta}. \tag{13}$$

The terms $-a$ and β are again the two roots of the quadratic equation

$$f(Q) = .5\sigma^2 Q(Q-1) + (\delta_W - \delta_R)Q - \delta_W = 0,$$

and they equal

$$0.5 - (\delta_W - \delta_R)/\sigma^2 \pm \surd\{[(\delta_W - \delta_R)/\sigma^2 - 0.5]^2 + 2\delta_W/\sigma^2\}, \tag{14}$$

with the effective variance term defined in equation (9). Now we can solve for the parameters A and B, and for the upper and lower trigger points R_H and R_L, and thus value the claim in both the idle and the active modes.

As shown in Dixit (1989a) and Dixit and Pindyck (1994, pp. 213–229), the claim value and the width of the hysteresis zone are increasing in the variance. Noting the dependence of equation (9) on the terms in equations (6), that variance is a decreasing function of the correlation between revenues and costs $\rho_{AR,AW}$, the correlation between revenues and the exchange rate for costs $\rho_{AR,EW}$, the correlation between the costs and the exchange rate for the revenues $\rho_{ER,AW}$, and the correlation between the two exchange rates $\rho_{ER,EW}$. The effective variance is also increasing in the correlation between the costs and the exchange rate for the costs $\rho_{AW,EW}$, and in the correlation between the revenues and the exchange rate for the revenues $\rho_{AR,ER}$. In the above, all exchange rates and correlations are calculated from the home country's perspective. We must note that all riskless interest rates appear to vanish, although they are practically embedded in the dividend yields δ_W and δ_R. Our three-country model also allows directly for the two- or the one-country model as a special case, when the operation cost W is stochastic. When W is fixed (constant, and defined not in foreign but in home currency), the riskless rate of the home country can be recovered if we note that, by definition, the dividend yield (see McDonald and Siegel, 1984, 1986) equals the required return on W minus the expected growth rate, in which case we get $\delta_W = R_W - m_W = r - 0 = r$. In the next section we first provide a direct application of this extended version of the hysteresis model, and then an implementation in a multistage setting that captures interactions between investment and operating decisions.

3. Applications

We discuss two applications of the hysteresis model for a multinational corporation facing multiple uncertainties. We first demonstrate the valuation of a productive operation with stochastic revenues and costs in a three-country setting; thus, exchange rate risks affect both cash flows. Then, we provide a richer framework that captures interactions between wait-to-invest and operating options. The (compound) option to the rights to that operation is an option to wait-to-invest, and its underlying asset is the productive operation, valued endogenously with the use of the hysteresis model. Numerical simulations demonstrate the significance of the volatility and the correlation structure on valuation and optimal investment/operating decisions.

Brennan and Schwartz (1985) and Dixit (1989a) examined the importance of the hysteresis model in relation to a single cash flow uncertainty, whereas Pindyck (1993) presented a valuation model of operations with flexibility where costs are the main source of uncertainty. Dixit (1989b), Bell (1995), and Mello et al. (1995) investigated the importance of currency fluctuations as the single source of uncertainty within the hysteresis framework. Barham et al. (1998) identified the role of hysteresis effects in explaining the strategic behavior in the natural resources extracting industries in the Americas, and Lund (1999) identified the need for operational flexibility in the exploitation of the Norwegian oil reserves. Moel and Tufano (1999) provided emperical evidence that both revenue and cost uncertainty contribute to hysteresis. Our extension of the (operational flexibility) hysteresis model naturally incorporates revenue, cost and exchange rate uncertainty in an international setting. Our model with the resulting hysteresis zone is clearly demonstrated in the *first* application. Consider the corporation (in the home country) that owns the option to operate in a second country where it is incurring continuous costs (wages, etc.) denominated in foreign currency; at the same time, this operation is generating revenues in a third country, again denominated in foreign currency. The parameter values for the base case appear at the bottom of Table 1. With the use of equations (6) and (9) we get the effective standard deviation $\sigma = 0.2208$. Through the numerical solution described in the Appendix we get the upper and lower trigger points R_H and R_L that appear in Table 1, the values of the active (through equation 12) and the idle (through equation 13) operations V_1 and V_0, and the optimal operating decision. The first four lines remind us of the basic insights gained from the original hysteresis model, while the rest demonstrate the impact of the correlation and the volatility structure of our extended version of the hysteresis model on the optimal operating decisions and the option value.

Table 1. A multinational operation with switching options and multiple uncertainties

	Upper Trigger R_H	Low Trigger R_L	Active Value (V_1), Less Opening Cost k	Idle Value (V_0), Less Closing Cost l	Optimal Operating Decision
Base Case	$2.3752W$	$0.5192W$	10.3514, −5	5.7818, −2	Do not Switch
$R = 1.5$	$2.3752W$	$0.5192W$	5.5546, −5	2.7475, −2	Do not Switch
$R = 0.6$	$2.3752W$	$0.5192W$	−1.6272, −5	0.2569, −2	Do not Switch
$k = 2W, l = 1W$	$1.8004W$	$0.6194W$	10.5022, −2	—	Active Mode
$\rho_{AR,AW} = \rho_{ER,EW} = 0.45$, $\rho_{AR,EW} = \rho_{ER,AW} = 0.25$, $\rho_{AR,ER} = \rho_{AW,EW} = -0.25$	$1.7754W$	$0.6764W$	10.0035, −5	—	Active Mode
$\rho_{AR,AW} = \rho_{ER,EW} = 0.45$, $\rho_{AR,EW} = \rho_{ER,AW} = 0.25$, $\rho_{AR,ER} = \rho_{AW,EW} = -0.25$, and $R = 0.6$	$1.7754W$	$0.6764W$	—	0.0026, −2	Idle Mode
$\sigma_{EW} = \sigma_{ER} = 0$ (No Currency Risk)	$2.2144W$	$0.5512W$	10.2015, −5	5.3928, −2	Do not Switch
$\sigma_{EW} = \sigma_{ER} = 0.25$ (High Currency Risk)	$3.0160W$	$0.4311W$	11.1222, −5	7.4649, −2	Do not Switch
$\rho_{AR,ER} = \rho_{AW,EW} = -0.50$	$1.9722W$	$0.6117W$	10.0489, −5	—	Active Mode
$\rho_{AR,ER} = \rho_{AW,EW} = 0.50$	$2.6214W$	$0.4793W$	10.6275, −5	6.4337, −2	Do not Switch
$\rho_{ER,EW} = -0.50$	$2.5171W$	$0.4951W$	10.5053, −5	6.1545, −2	Do not Switch

Note: For the base case parameter values used are: continuous revenues $R = 2$, continuous costs $W = 1$, opening costs $k = 5W$, closing costs $l = 2W$, dividend yield for revenues and costs $\delta_{AR} = \delta_{AW} = 0.10$; the standard deviations $\sigma_{AR} = 0.15$ for the revenues, $\sigma_{AW} = 0.15$ for the costs, $\sigma_{ER} = 0.1$ for the exchange rate with the country where revenues are generated, and $\sigma_{EW} = 0.1$ for the exchange rate with the country where the costs are incurred; and the correlations $\rho_{AR,AW} = 0.25$, $\rho_{AR,EW} = 0.05$, $\rho_{ER,AW} = 0.05$, $\rho_{ER,EW} = 0.25$, $\rho_{AR,ER} = 0.05$, and $\rho_{AW,EW} = 0.05$. "Do not switch" demonstrates the existence of hysteresis: if operation was in the active mode, it remains active, and if it was in the idle mode, it remains idle.

In the first line (the base case) the operation falls within the hysteresis zone. We see that if the project is idle, it should remain idle, since to receive a value of $V_1 = 10.3514$ we must pay $k = 5W = 5$ for a net benefit of $5.3514 < 5.7818 = V_0$. The project will remain idle until revenues R exceed $2.3752W$. Similarly, if the project is active it should remain active. Similar results we observe when $R = 0.6$. The value of the active project is -1.6272 and the value of the idle project is 0.2569. Again, if the project is idle it should remain idle. If it is active, it should remain active until revenues R drop below $0.5192W$, since to close management must incur cost $l = 2W = 2$ in order to receive 0.2569, and the active project value of $V_1 = -1.6272$ is preferred to switching off for a value of $0.2569 - 2 = -1.7431 = V_0$. Higher revenues ($R = 1.5$) leave the hysteresis zone unaffected and increase option values. In the fourth line, we see that lower switching costs provide a narrower hysteresis zone and also increase option values.

The hysteresis zone is affected by the correlation structure: as shown in the fifth and the sixth lines, the zone becomes substantially narrower when the correlations between each capital flow and the relevant exchange rate are lower and all the rest are higher (since now the effective standard deviation $\sigma = 0.0758$). For that correlation structure, with revenues $R = 2$ the project should be active, and if revenues were $R = 0.6$ the project should be idle, in contrast with the first and third lines, respectively, were the project would fall within the hysteresis zone. We can observe similar results in the last three lines, where selected correlations are changed to further demonstrate the impact on the hysteresis zone. Although the direction of the impact of individual volatilities is in general ambiguous, the volatility structure can widen or narrow this zone significantly. See our example without currency risk ($\sigma_{ER} = 0$ and $\sigma_{EW} = 0$) where the hysteresis zone becomes narrower, and with high currency risk ($\sigma_{ER} = 0.25$ and $\sigma_{EW} = 0.25$) where the zone widens considerably. We have see that the correlation and volatility structure (and the switching costs k and l) significantly affect the zone of hysteresis and the optimal *operating* decision.

In the model discussed so far we have assumed that the flexible operation is already owned and practically ready to be switched on and off. But often, before this is the case, exploration/development costs must be incurred, which will add a *compound* option framework (see Geske, 1977; Carr, 1988) to the analysis. This sequential decision model is studied numerically in Paddock et al. (1988) with the assumption of no operational flexibility, as an extension of the McDonald and Siegel (1986) option to wait-to-invest. However, interactions of operational flexibility and investment decisions are important (for general discussions see Trigeorgis, 1993, 1996; for extensive studies of such interactions in the absence of hysteresis see Martzoukos and Trigeorgis, 1998). In the *second* application we extend the model to such a compound option framework, thus incorporating both operational flexibility and optimal investment decision-making in the presence of revenue, cost, exchange rate uncertainty, and hysteresis-inducing switching costs. We demonstrate the sensitivity of optimal investment decisions on parameter values, and also show large deviations between the option value of such investments and the NPV of cash flows (thus explaining the observed practice of "over-paying" for such investment rights; see the recent *Business Week* article, "Exploiting Uncertainty"). Think of a multinational corporation that does not yet own, but has (or considers the purchase of) the option to acquire the rights to the flexible operation. Since that operation already includes the option to switch back and forth with switching costs k and l as defined earlier, the option to that operation is a *compound* one (see Figure 1). This compound option can be exercised at some cost X up to time T (an American-type option), or at exactly time T (a European-type option). The hysteresis model provides the value of the productive entity with the flexibility to switch between modes of operation, whereas the option to acquire the rights to that entity is a real option to wait-to-invest where the underlying asset must be calculated endogenously using the hysteresis model.

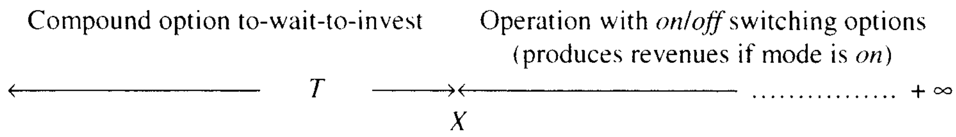

Figure 1. The two-stage investment option with hysteresis.

For the base case we use the same parameter values as in the first application. We need to make the additional assumption that the cost X of exercising the real option is proportional to the continuous costs W of operation. Thus, X is stochastic and dependent on the same state-variable(s) as W. We will assume that the investment opportunity expires at $T = 2$ years, and that if the investment option is exercised the operation will become active for the first time (and switching costs k will be incurred). We implement a binomial lattice with 200 steps and value both the European and American options on that productive operation. For the European investment option the hysteresis model is employed 201 times at maturity. For the American option the hysteresis model is employed 20301 times up to maturity to test for early exercise. Of course, we need to calculate the hysteresis band only once, and then apply the model to price the option for different values of the underlying state-variables. In such applications, the efficiency of the approach we develop in this paper is indeed critical. The numerical results for the value of investing now (acquiring the operation with the embedded option to switch on and off optimally), the European investment option (without optimal investment timing), the American investment option (with optimal investment timing), and the optimal investment decision are shown in Table 2. The results that can be contrasted with the readily calculated NPV ($= R/\delta_R - W/\delta_W - X - k$), and the impact of the correlation structure and the yearly revenues on optimal investment decision are shown. At time zero the value of the productive option is 10.3514, but in addition to the cost $X = 2$, $k = 5$ would have to be paid if it should get into the active mode. Since the American investment option of 4.0974 is greater than acquiring the rights immediately and opening the operation at a cost of $10.3514 - (X + k) = 3.3514$, the optimal investment decision is to wait. The first three lines clearly demonstrate that the NPV criterion can be grossly misleading, since it ignores both options to acquire the operation at optimal timing and to operate optimally.

If we use the correlation structure with an effective standard deviation $\sigma = 0.0758$ (fourth line), we get a European *compound* option value of 2.5064. But since the American option value of 3.0035 equals the value of the operation if we invest immediately (after costs X and k are paid), investing (not waiting) is the optimal decision. In the last two lines for revenues $R = 2.6$, a change in the correlation structure from the base case to $\rho_{AR,ER} = \rho_{AW,EW} = 0.50$ shifts the investment decision from "invest" to "wait". This change in the optimal investment policy was not possible for revenues $R = 3$, in which case the investment option was sufficiently in-the-money and it should be exercised for either correlation structure. Even if the optimal investment decision stays the same, the value of the investment rights (the compound option) can be affected significantly with a change in the correlation structure, as the previous three lines indicate. Similar effects can be observed with changes in the volatility structure, where the case of high currency risk gives option values almost double the case of no currency risk.

4. Summary and conclusions

This paper has demonstrated the hysteresis model of investment extended to the case of stochastic revenues and costs, and where both cash flows are generated in different foreign countries with stochastic exchange rates. If one of these cash flows occurs in the home

Table 2. Value of the (real) investment option to a multinational operation

	Value of Investing Now	Investment Option without Optimal Timing	Investment Option with Optimal Timing	Optimal Investment Decision
Base Case (NPV = 3)	10.3514 − 5 − 2 = 3.3514	3.8512	4.0974	Wait
$R = 1.5$ (NPV = −2)	5.5546 − 5 − 2 = −1.4454	1.3325	1.3571	Wait
$R = 0.60$ (NPV = −11)	−1.6272 − 5 − 2 = −8.6272	0.0018	0.0018	Wait
$\rho_{AR,AW} = \rho_{ER,EW} = 0.45$, $\rho_{AR,EW} = \rho_{ER,AW} = 0.25$, $\rho_{AR,ER} = \rho_{AW,EW} = -0.25$.	10.0035 − 5 − 2 = 3.0035	2.5064	3.0035	Invest
$\sigma_{EW} = \sigma_{ER} = 0$ (No Currency Risk)	10.2015 − 5 − 2 = 3.2015	3.4074	3.6681	Wait
$\sigma_{EW} = \sigma_{ER} = 0.25$ (High Currency Risk)	11.1222 − 5 − 2 = 4.1222	5.5772	5.8347	Wait
$\rho_{AR,ER} = \rho_{AW,EW} = -0.50$	10.0489 − 5 − 2 = 3.0489	2.8091	3.1392	Wait
$\rho_{AR,ER} = \rho_{AW,EW} = 0.50$	10.6275 − 5 − 2 = 3.6275	4.5362	4.7792	Wait
$\rho_{ER,EW} = -0.50$	10.5053 − 5 − 3 = 3.5053	4.2474	4.4899	Wait
$R = 3$ (NPV = 13)	20.1847 − 5 − 2 = 13.1847	10.9866	13.1847	Invest
$\rho_{AR,ER} = \rho_{AW,EW} = 0.50$ $R = 3$ (NPV = 13)	20.3857 − 5 − 2 = 13.3857	11.4261	13.3857	Invest
$R = 2.6$ (NPV = 9)	16.2318 − 5 − 2 = 9.2318	7.9444	9.2318	Invest
$\rho_{AR,ER} = \rho_{AW,EW} = 0.50$ $R = 2.6$ (NPV = 9)	16.4580 − 5 − 2 = 9.4580	8.4970	9.5232	Wait

Note: Parameter values for the productive operation with switching options are as in the base case in Table 1. The cost X of exercising the investment option is $2W = 2$. The investment can be deferred for $T = 2$ years (option maturity). We employ a 200-steps lattice. Investment option with optimal timing implies an American option with the early exercise feature.

country, or if both occur in the same foreign country, the model becomes a special case of the more general framework. If either of the cash flows is constant in a foreign country, the model is again a special case of the general one, but with the riskless rate of that foreign country replacing the relevant dividend yield. A critical assumption was made that the costs of switching between the idle and the active states are constant fractions of the costs. Our solution for the case of such higher dimensionality provides for more realistic applications.

Two trigger points define the hysteresis zone, which is affected by the effective variance. This variance decreases when the following correlations increase: the correlation between the revenues AR and the costs AW; the correlation between the exchange rates ER and EW; the correlation between AR and EW; and the correlation between ER and AW. The variance increases when the following correlations are higher: the correlation between AR and ER; and between AW and EW. The model assumes partial irreversibility of investment. The interaction of uncertainties in the context of complete irreversibility and N underlying assets (but in the absence of hysteresis) can be found in Martzoukos (1997).

An efficient numerical approach to solve the general hysteresis problem under infinite horizon with (or without) foreign exchange uncertainty is demonstrated in the Appendix. Such efficiency is instrumental for the solution of the more general multistage problems with interacting investment and operating options that we demonstrated. Assuming that all costs to enter the next stage of multistage options are constant fractions of the continuous operating costs, the dimensionality of the investment option is, for practical purposes, reduced from four to one. In that more general framework we considered a two-stage problem partition. In the first stage, the decision to invest in the productive operation must be optimally made. Once such a decision has been positive, the optimal operating decisions must be made continuously, with hysteresis affecting the optimal switching points between the active and idle modes.

Appendix: An efficient numerical method for solving the infinite horizon hysteresis model

We propose a specific method to solve the general hysteresis model. The specific form of the solution is very efficient because we reduce the dimensionality of the problem from a nonlinear system of four equations with four unknowns, to a nonlinear system of two equations with two unknowns. It is important to note that we restrict to the two unknowns (the two trigger points) for which we have information, thus considerably reducing the numerical search process. We know that the upper trigger point is restricted to be equal to or greater than W. Also the lower trigger point is restricted to be less than or equal to W, and greater than zero. In addition, the specific solution form that we implement is used in the first section of this paper to demonstrate the homogeneity property of the hysteresis model. The solution method follows.

We must solve the following system of four equations, derived from the value matching and the smooth pasting conditions

$$\begin{aligned} &BR_H^{\beta} - AR_H^{-a} - R_H/\delta + W/r = -k, \\ &BR_L^{\beta} - AR_L^{-a} - R_L/\delta + W/r = l, \\ &\beta BR_H^{\beta} + aAR_H^{-a} - R_H/\delta = 0, \\ &\beta BR_L^{\beta} + aAR_L^{-a} - R_L/\delta = 0. \end{aligned} \tag{A1}$$

The system of equations (A1) is transformed to a system of two equations. Note that from the third and the fourth equations we get

$$B = \left[R_H^{1-\beta}/\delta - aAR_H^{-a-\beta}\right]/\beta,$$

and

$$A = R_L^{1+a}/(a\delta) - R_L^{\beta}\left[R_H^{1-\beta}/\delta - aAR_H^{-a-\beta}\right]/\left(aR_L^{-a}\right),$$

thus we eliminate A and B from the RHS using

$$A = \left[R_L^{1+a}/(a\delta) - R_L^{\beta+a}R_H^{1-\beta}/(\delta a)\right]/\left[1 - R_L^{\beta+a}R_H^{-a-\beta}\right], \tag{A2}$$

and

$$B = R_H^{1-\beta}/(\beta\delta) - \left[R_L^{1+a}R_H^{-a-\beta}/(\beta\delta) - R_L^{a+\beta}R_H^{1-a-2\beta}/(\beta\delta)\right]/\left[1 - R_L^{\beta+a}R_H^{-a-\beta}\right]. \tag{A3}$$

By substituting back into the first two equations of (A1), we get

$$\begin{aligned} &R_H[1/(\beta\delta) - 1/\delta] - \left\{\left[aR_L^{1+a}R_H^{-a} - aR_L^{a+\beta}R_H^{1-a-\beta} + \beta R_H^{-a}R_L^{1+a}\right.\right.\\ &\quad \left.\left.- R_L^{a+\beta}R_H^{1-a-\beta}\beta\right]/\left[a\beta\delta\left(1 - R_L^{a+\beta}R_H^{-a-\beta}\right)\right]\right\} + W/r + k = 0, \end{aligned} \tag{A4}$$

and

$$\begin{aligned} &R_L^{\beta}R_H^{1-\beta}/(\beta\delta) - R_L/\delta - \left\{\left[aR_L^{1+a+\beta}R_H^{-\beta-a} - aR_L^{a+\beta}R_H^{1-a-2\beta} + \beta R_L\right.\right.\\ &\quad \left.\left.- R_L^{\beta}R_H^{1-\beta}\beta\right]/\left[a\beta\delta\left(1 - R_L^{a+\beta}R_H^{-a-\beta}\right)\right]\right\} + W/r - l = 0. \end{aligned} \tag{A5}$$

To implement a two-dimensional, Newton-Raphson scheme we need to calculate the [2 × 2] Jacobian matrix of the analytic derivatives. The components of this matrix are denoted as $J_0'(R_H)$, $J_1'(R_H)$, $J_0'(R_L)$, $J_1'(R_L)$. The term in parenthesis denotes the variable of partial differentiation, and the subscript of zero or one implies that this corresponds to the equation (A4) or equation (A5), respectively. We get

$$\begin{aligned} J_0'(R_H) = {}& 1/(\beta\delta) - 1/\delta + \left\{\left[a^2R_L^{1+a}R_H^{-1-a} + a\beta R_H^{-1-a}R_L^{1+a}\right.\right.\\ &+ a(1-a-\beta)R_L^{a+\beta}R_H^{-a-\beta} + (\beta - \beta a - \beta^2)R_L^{a+\beta}R_H^{-a-\beta}\left.\right]/\left[\left(1\right.\right.\\ &\left.\left.\left.- R_L^{a+\beta}R_H^{-a-\beta}\right)a\beta\delta\right]\right\} + \left\{\left[aR_L^{1+a}R_H^{-a} - aR_L^{a+\beta}R_H^{1-a-\beta} + \beta R_H^{-a}R_L^{1+a}\right.\right.\\ &\left.\left.- \beta R_L^{a+\beta}R_H^{1-a-\beta}\right]\left[(a+\beta)R_L^{a+\beta}R_H^{-a-\beta-1}\right]/\left[\left(1 - R_L^{a+\beta}R_H^{-a-\beta}\right)^2 a\beta\delta\right]\right\}, \end{aligned}$$

$$\begin{aligned} J_0'(R_L) = {}& -\left\{\left[a(1+a)R_L^{a}R_H^{-a} - a(\beta+a)R_H^{-1-a-\beta}R_L^{-1+\beta+a} + \beta(1+a)R_L^{a}R_H^{-a}\right.\right.\\ &\left.\left.+ (a+\beta)R_L^{a+\beta-1}R_H^{1-a-\beta}\right]/\left[\left(1 - R_L^{a+\beta}R_H^{-a-\beta}\right)a\beta\delta\right]\right\}\\ &- \left\{\left[aR_L^{1+a}R_H^{-a} - aR_L^{a+\beta}R_H^{1-a-\beta} + \beta R_H^{-a}R_L^{1+a}\right.\right.\\ &\left.\left.- \beta R_L^{a+\beta}R_H^{1-a-\beta}\right]\left[(a+\beta)R_L^{-1+a+\beta}R_H^{-a-\beta}\right]/\left[\left(1 - R_L^{a+\beta}R_H^{-a-\beta}\right)^2 a\beta\delta\right]\right\}, \end{aligned}$$

$$
\begin{aligned}
J_1'(R_H) = {} & (1-\beta)R_L^{\beta}R_H^{-\beta}/(\beta\delta) - \big\{\big[a(-a-\beta)R_L^{1+a+\beta}R_H^{-1-a-\beta} \\
& - a(1-2\beta-a)R_H^{-a-2\beta}R_L^{2\beta+a} - \beta(1-\beta)R_L^{\beta}R_H^{-\beta}\big]/\big[\big(1-R_L^{a+\beta}R_H^{-a-\beta}\big)a\beta\delta\big]\big\} \\
& - \big\{\big[aR_L^{1+a+\beta}R_H^{-a-\beta} - aR_L^{a+2\beta}R_H^{1-a-2\beta} + \beta R_L \\
& - \beta R_L^{\beta}R_H^{1-\beta}\big]\big[(-a-\beta)R_L^{a+\beta}R_H^{-a-\beta-1}\big]/\big[\big(1-R_L^{a+\beta}R_H^{-a-\beta}\big)^2 a\beta\delta\big]\big\},
\end{aligned}
$$

$$
\begin{aligned}
J_1'(R_L) = {} & R_L^{\beta-1}R_H^{1-\beta}/\delta - 1/\delta - \big\{\big[a(1+a+\beta)R_L^{a+\beta}R_H^{-a-\beta} \\
& - a(2\beta+a)R_H^{1-a-2\beta}R_L^{2\beta+a-1} + \beta - \beta^2 R_L^{\beta-1}R_H^{1-\beta}\big]/\big[\big(1-R_L^{a+\beta}R_H^{-a-\beta}\big)a\beta\delta\big]\big\} \\
& - \big\{\big[aR_L^{1+a+\beta}R_H^{-a-\beta} - aR_L^{a+2\beta}R_H^{1-a-2\beta} + \beta R_L \\
& - \beta R_L^{\beta}R_H^{1-\beta}\big]\big[(a+\beta)R_L^{a+\beta-1}R_H^{-a-\beta}\big]/\big[\big(1-R_L^{a+\beta}R_H^{-a-\beta}\big)^2 a\beta\delta\big]\big\}.
\end{aligned}
$$

The above elements of the Jacobian $\boldsymbol{J}$ are utilized in a two-dimensional, Newton–Raphson method to solve the system of the two nonlinear equations $\boldsymbol{F}(R_H, R_L) = 0$. For a small change in $\boldsymbol{R}$ equal to $d_{\boldsymbol{R}}$, we get

$$F(\boldsymbol{R} + d_{\boldsymbol{R}}) = F(\boldsymbol{R}) + J d_{\boldsymbol{R}}$$

where higher order terms of the two-dimensional Taylor series expansion are neglected. We set the two equations equal to zero, thus we get $-F(\boldsymbol{R}) = \boldsymbol{J}d_{\boldsymbol{R}}$. The terms $d_{\boldsymbol{R}}$ are derived from the expansion of the above matrix equation to the two equations

$$
\begin{aligned}
J_0'(R_H)d_H + J_0'(R_L)d_L &= -F_0, \\
J_1'(R_H)d_H + J_1'(R_L)d_L &= -F_1,
\end{aligned}
$$

which are solved for d_H, and d_L

$$
\begin{aligned}
d_L &= [-F_1 + F_0 J_1(R_H)/J_0'(R_H)]/[J_1'(R_L) - J_1'(R_H)J_0'(R_L)/J_0'(R_H)], \\
d_H &= [-F_0 - J_0'(R_L)d_L]/J_0'(R_H).
\end{aligned}
$$

We use $d_{\boldsymbol{R}}$ to iterate according to $\boldsymbol{R}_1 = \boldsymbol{R}_0 + d_{\boldsymbol{R}}$ until convergence to required numerical accuracy. Note that the search for R_H and R_L is restricted to $R_H \geq W$ and $W \geq R_L \geq 0$. From R_H and R_L, we calculate the parameters A and B, and subsequently we can price the operation in the active and in the idle modes.

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References

Barham, B. L., J. P. Chavas and O. T. Coomes, “Sunk Costs and the Natural Resource Extraction Sector: Analytical Models and Historical Examples of Hysteresis and Strategic Behavior in the Americas.” *Land Economics* 74, 429–448, (1998).

Bell, K. G., “Volatile Exchange Rates and the Multinational Firm: Entry, Exit, and Capacity Options,” in Trigeorgis (Ed), *Real Options in Capital Investment: Models, Strategies, and Applications,* Westport, Connecticut: Praeger, 1995.

Black, F. and M. Scholes, “The Pricing of Options and Corporate Liabilities.” *Journal of Political Economy* 81, 637–659, (1973).

Breeden, D. T., “An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities.” *Journal of Financial Economics* 7, 265–296, (1979).

Brennan, M. J., “The Price of Convenience and the Valuation of Commodity Contingent Claims,” in D. Lund and B. Öksendal (Eds), *Stochastic Models and Option Values,* New York: North-Holland, 1991.

Brennan, M. J. and E. Schwartz, “Evaluating Natural Resource Investments.” *Journal of Business* 58, 135–157, (1985).

Business Week “Exploiting Uncertainty.” June 7, 118–124, (1999).

Carr, P., “The Valuation of Sequential Exchange Opportunities.” *Journal of Finance* XLIII, 1235–1256, (1988).

Constantinides, G., “Market Risk Adjustment in Project Valuation.” *Journal of Finance* 33, 603–616, (1978).

Cox, J., J. Ingersoll and S. Ross, “An Intertemporal General Equilibrium Model of Asset Prices.” *Econometrica* 53, 363–384, (1985).

Crousillat, E. and S. H. Martzoukos, *Decision Making under Uncertainty—An Option Valuation Approach to Power Planning,* IENED Working Paper #39, The World Bank, (1991).

Dixit, A., “Entry and Exit Decisions under Uncertainty.” *Journal of Political Economy* 97, 620–638, (1989a).

Dixit, A., “Hysteresis, Import Penetration, and Exchange Rate Pass-Through.” *Quarterly Journal of Economics* 104, 205–228, (1989b).

Dixit, A., “Intersectoral Capital Allocation Under Price Uncertainty.” *Journal of International Economics* 26, 309–325, (1989c).

Dixit, A., “Analytical Approximations in Models of Hysteresis.” *Review of Economic Studies* 58, 141–151, (1991).

Dixit, A., “Investment and Hysteresis.” *Journal of Economic Perspectives* 6, 107–132, (1992).

Dixit, A. and R. Pindyck, *Investment under Uncertainty*. Princeton NJ: Princeton University Press, 1994.

Garman, M. and S. W. Kohlhagen, “Foreign Currency Option Values.” *Journal of International Money and Finance* 2, 231–237, (1983).

Geske, R., “The Valuation of Corporate Liabilities as Compound Options.” *Journal of Financial and Quantitative Analysis* 12, 541–552, (1977).

Grabbe, O. J., “The Pricing of Call and Put Options on Foreign Exchange.” *Journal of International Money and Finance* 2, 239–253, (1983).

Harrison, J. M. and D. Kreps, “Martingales and Arbitrage in Multiperiod Security Markets.” *Journal of Economic Theory* 20, 381–408, (1979).

Harrison, J. M. and S. Pliska, “Martingales and Stochastic Integrals in the Theory of Continuous Trading.” *Stochastic Processes and Their Applications* 11, 313–316, (1981).

Hull, J., *Options, Futures, and other Derivatives,* Upper Saddle River, New Jersey: Prentice Hall, 1997.

Karatzas, I. and S. E. Shreve, *Brownian Motion and Stochastic Calculus,* Berlin, Germany: Springer-Verlag, 1991.

Kat, H. M. and H. N. M. Roozen, “Pricing and Hedging International Equity Derivatives.” *Journal of Derivatives* 2, 7–19, (1994).

Lund, M. W., “Real Options in Offshore Oil Field Development Projects.” Paper presented at the 3rd Annual Real Options Conference, Wassenaar/Leiden, The Netherlands, 1999.

McDonald, R. and D. Siegel, “Option Pricing when the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note.” *Journal of Finance* 39, 261–265, (1984).

McDonald, R. and D. Siegel, “The Value of Waiting to Invest.” *Quarterly Journal of Economics* 101, 707–727, (1986).

Malliaris, A. G. and W. Brock, *Stochastic Methods in Economics and Finance,* New-York: North-Holland, 1982.
Margrabe, W., "The Value of an Option to Exchange One Asset for Another." *Journal of Finance* 33, 177–186, (1978).
Martzoukos, S. H., *Multivariate Contingent Claims with Stochastic Exercise Price and Multiplicative Uncertainties,* SBPM Working Paper 96-42, George Washington University, (1997).
Martzoukos, S. H. and Lenos Trigeorgis, "Solving General Multi-Stage, Multi-Dimensional Investment Problems." Paper presented at the 2nd Annual Real Options Conference, Northwestern University, Chicago, 1998.
Mauer, D. and A. Triantis, "Interactions of Corporate Financing and Investment Decisions: A Dynamic Framework." *Journal of Finance* 49, 1253–1277, (1994).
Mello, A. S., J. E. Parsons and A. J. Triantis, "An Integrated Model of Multinational Flexibility and Financial Hedging." *Journal of International Economics* 39, 27–51, (1995).
Merton, R., "The Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4, 141–183, (1973a).
Merton, R., "An Intertemporal Capital Asset Pricing Model." *Econometrica,* 867–887, (1973b).
Moel, A. and P. Tufano, "When are Real Options Exercised? An Empirical Study of Mine Closings." Paper delivered at the 3rd Annual Real Options Conference, Wassenaar/Leiden, The Netherlands, (1999).
Paddock, J. L., D. R. Siegel and J. L. Smith, "Option Valuation of Claims on Real Assets: The Case of Offshore Petroleum Leases." *Quarterly Journal of Economics* 103, 479–508, (1988).
Pindyck, R., "Irreversibility, Uncertainty and Investment." *Journal of Economic Literature* 29, 1110–1152, (1991).
Pindyck, R., "Investments of Uncertain Cost." *Journal of Financial Economics* 34, 53–76, (1993).
Reiner, E., "Quanto Mechanics." *RISK* March, 59–63, (1992).
Trigeorgis, L., "The Nature of Option Interactions and the Valuation of Investments with Multiple Real Options." *Journal of Financial and Quantitative Analysis* 28, 1–20, (1993).
Trigeorgis, L., *Real Options: Managerial Flexibility and Strategy in Resource Allocation,* Cambridge: The MIT Press, 1996.