REAL OPTION GAMES WITH INCOMPLETE INFORMATION AND SPILLOVERS

by
Spiros H. Martzoukos and Eleftherios Zacharias

Working Paper 01-20

HERMES Center of Excellence
on Computational Finance & Economics
University of Cyprus
P.O. Box 20537, 1678 Nicosia, CYPRUS

HERMES Center of Excellence on Computational Finance & Economics

The HERMES Center on Computational Finance and Economics at the University of Cyprus has been selected by the European Commission as a European Center of Excellence in 2000. The Center faculty, graduate students, and visitors pursue a broad research agenda that focuses on optimal financial decision making from both the supply side (financial institutions) and the demand side (households and institutional investors). Emphasis is placed on the challenges created for both sides by the globalization and innovations of the financial markets, especially for the economies of pre-accession States as they move towards harmonization with European Union.

The work of the Center is divided in five major areas. The first deals with enterprise wide risk management and the development of both innovative methodologies and decision support tools. The second deals with the saving and borrowing behavior of households and their portfolio choices. The third deals with empirical studies of capital markets and the information they reveal for the prediction of bankruptcy events and the management of credit risk. The fourth deals with real options, their valuation, and their use in coping with an uncertain world. The fifth deals with issues surrounding the performance of financial institutions in delivering quality services.

Data-driven theoretical modeling, empirical analysis, and the use of computations are the cornerstones of the disciplinary approaches of the Center's activity, that brings together researchers from finance, economics, accounting, management sciences, mathematics and computer science. Research is basic, yet relevant to this important sector of economic activity.

The Center fosters a community of scholars, practitioners, and graduate students whose interests support the mission of the Center. Close collaboration with industry ensures that the Center's research remains not only cutting-edge in pursuit of academic excellence, but is also relevant to financial institutions, in their quest for competitive excellence. If you would like to know more about the Center or join in its activities please let us know of your interest and visit our Web site at http://www.hermes.ucy.ac.cy/

Stavros A. Zenios

Stavros A. Zenios
Director

Real Option Games with Incomplete Information and Spillovers

Spiros H. Martzoukos and Eleftherios Zacharias

School of Economics and Management

University of Cyprus - Nicosia

May 2001, this version July 2001 (edited November 2001)

JEL classification: G13, G31; C72; L00

Keywords: Real Options, Incomplete Information and Learning, R&D Coordination, 2-Stage Games

This work was partly completed while the first author visited the Centre for Financial Research (Judge Institute of Management, Cambridge U.) and is grateful for the hospitality. He wishes to acknowledge support from the Hermes Center of Excellence in Computational Finance and Economics at the U. of Cyprus. Comments by seminar participants at the *Managing Uncertainty* Program (I. Newton Institute at Cambridge U., August 2001), at the University of Cyprus, and by G. Deltas, M. Dempster, D. Goodman, M. Haliasos, S. Klerides, B. Lambrecht, and T. Stengos, are greatly appreciated.

Address correspondence:

Spiros H. Martzoukos, Assistant Professor
Department of Public and Business Administration,
School of Economics and Management – University of Cyprus
P.O.Box 20537, CY 1678 Nicosia, CYPRUS. Tel.: 357-2-892474, Fax: 357-2-892460.
Email: baspiros@ucy.ac.cy

Real Option Games with Incomplete Information and Spillovers

Abstract

We discuss in a duopolistic game theoretic context managerial intervention directed towards value enhancement in the presence of uncertainty and spillover effects. Two firms have monopoly power over real investment opportunities, and before making the irreversible investment decisions, they have options to enhance value by doing more R&D and/or acquiring more information. Due to spillovers, managers act strategically by optimizing their behavior, conditional on the actions of their counterpart. They face two decisions that are solved for interdependently in a two-stage game. The first decision is: what is the optimal effort for a given level of the spillover effects and the cost of information acquisition? The second decision is: what is the optimal level of coordination between them?

Introduction

We discuss in a game theoretic context managerial intervention directed towards value enhancement in the presence of uncertainty and spillover effects. Two firms face real investment options. Embedded in these are (optional) actions that allow firms to enhance value of their prospects directly (through R&D that improves product attributes or reduces costs, advertisement, etc.) or indirectly through information acquisition (exploratory drilling, marketing research, etc.) Due to spillovers, each firm´s action affects the other firm. In this framework both firms can act strategically and take advantage of the positive spillovers (or take pre-emptive action against the negative spillovers). In equilibrium, the degree of coordination can be higher or lower. The implementation of strategy by each firm can either be implicit, or explicit, i.e., by forming a research joint venture. Each firm must thus decide: (a) How much to spend for such (R&D, etc.) actions, given the spillover effects and (b) How much from this effort to share with its counterpart if this level can vary. For the first (a tactical) decision we allow for a continuous set of alternatives, whereas for the second (a strategic) one we only allow for a discrete set of choices. Let for example a pharmaceutical and a chemical firm trying to enhance the value of their investment prospects, by taking R&D actions to develop new technologies. Each part can learn at least part of the results of the effort of the other for free even if both act on their own. However the two firms will optimize the value of their investment options if they strategically determine the amount of R&D they share (in forming for example a research joint venture and deciding the degree of coordination that will take place and the type of research they will engage in). Another example would be two oil firms that own rights in adjacent oil fields. Knowledge resulting from exploratory drilling could be shared benefiting both.

We adopt the real options framework from the financial economics literature and connect it with the research joint ventures literature of industrial organization. The literature on real options or otherwise irreversible investments under uncertainty (see Dixit and Pindyck, 1994, and Trigeorgis, 1996) examines the value of flexibility in investment and operating decisions under uncertainty where the traditional net present value (NPV) approach fails. Although the importance of learning actions like exploration, experimentation, and R&D was recognized early on (e.g., Roberts and Weitzman, 1981), the real options literature has paid little attention to management´s ability to intervene in order to acquire more information and/or enhance value. Majd and Pindyck (1989), and Pennings and Lint (1997) examine real options with passive learning, while Childs, et al. (1999), and Epstein, et al. (1999) use a filtering approach towards learning. Recently, Sundaresan (2000) emphasizes the need for adding an incomplete information framework to real options valuation problems. Although many state-variables are usually treated as observable, it is often more realistic to assume that they are only estimates of quantities that will be actually observed or realized later. Martzoukos (2000) considers true value as a random variable with a known probability distribution. He examines real options with controlled jumps of random size (random controls) to model intervention of management as intentional actions with uncertain outcome (learning). He assumes that sequential actions are independent of each other. Martzoukos and Trigeorgis (2000) extend this framework to one with path-dependent actions in order to explain the learning behavior of the (single) firm. They demonstrate that activating learning actions before an investment decision is made, is the solution to an optimization problem that actually captures the trade-off between learning early and paying a cost for it, or leaving information to reveal itself ex-post at a potential cost (of exercising seemingly profitable investment options that actually have a negative NPV, or leaving seemingly unprofitable investment options to expire unexercised when they actually have a positive NPV). We adopt this random controls methodology to examine in a game theoretic framework the behavior of two firms in the existence of spillovers.

Real options papers in a game-theoretic context are Grenadier (1996) with strategic option exercise in real asset markets, where development might be sequential; Williams (1993) with symmetric and simultaneous exercise strategies for real estate developers; and Smit and Ankum (1993) with an exogenously determined set of alternative corporate strategies. In this paper the interaction that results in a game theoretic framework comes from the existence of spillovers (in marketing research, R&D, etc.), and the incomplete information from the uncertain outcome of the firm´s actions. The importance of intra-industry spillovers has been documented in the literature. Foster and Rosenzweig (1995) for example emphasize the importance of learning spillovers in agriculture, and Carey and Bolton (1996) argue that collusion in advertisement can be successful due to significant spillovers from generic advertising. Spillover effects are significant even among different sectors, as discussed in Bernstein and Nadiri (1988) and Jaffe and Trajtenberg (1998), and among different countries, as postulated in Thompson (2000). Branstetter (1996) investigates the US and Japan to see whether technological externalities are more international or intranational in scope and finds more support for the latter, while Johnson and Evenson (1999) investigating 14 less developed or developing countries find empirical evidence that both international and interindustrial R&D spillovers can be significant.

Managers may have some control over spillover effects by deciding on the extent of R&D coordination without necessarily colluding in the product markets (i.e., semi-collusion to use the term in Fershtman and Gandal, 1994, Brod and Shivakumar, 1999, etc.) They can decide their strategy for the optimal degree of coordination by taking properly into consideration the benefits of R&D/learning investments. Different strategies of coordination include research joint ventures (RJV) and non-equity co-development (COD) according to the access the firms have to the innovation (see for example Tao and Wu, 1997). The seminal theoretic model of R&D choice in the presence of spillovers under various structures in the product market is d´Aspremont and Jacquemin (1988). Here the degree of spillover is the same. In Kamien et.al. (1992) the degree of spillovers varies. Firms can either coordinate in R&D or engage in information sharing by forming an RJV. Poyago-Theotoky (1999) recommends that firms endogenously determine the optimal degree of spillovers. In all papers firms operate in the same product market, and authors search for symmetric equilibria. In our paper, the choice of the level of spillovers is endogenously determined. For purposes of analytic tractability, we focus on the case where the firms do not compete in the product market (i.e., they have monopoly power over their investment option). Since firms operate in different product markets, we allow for asymmetric equilibria in the degree of spillovers between the firms.

The paper proceeds as follows. We first demonstrate the model for controlled learning, the framework for the tactical (resource allocation) decision with continuous choices, and the strategic (coordination) decision with a discrete set of choices. The next section presents an application with numerical results and discussion, and the final section concludes. In what follows, pre-investment management actions are intended to either enhance value directly (traditional R&D for attribute improvements or cost reduction, but also advertisement, etc.) or, in the presence of incomplete information about unobservable variables that influence the decision-making process, enhance value indirectly with efforts to resolve or at least reduce uncertainty (i.e., exploration, pharmaceutical experimentation, but also marketing research, etc.)

The real option game with incomplete information, multiplicative controls, and spillovers.

We consider costly R&D (control) actions of which the outcome will be observed instantaneously. We classify them into two types, pure *learning* control actions with the sole purpose of information acquisition, and *impact* control actions with a direct value enhancement (or similarly a cost reduction) purpose. Ex ante we simply know the probability distribution of the outcome, thus we call them random controls. The type of impact control is the most obvious one. Advertisement, process improvement, product attribute enhancement, etc., are actions that result directly in adding value, increasing sales volume and/or price per item, enhancing market share, or reduce production costs. In contrast, pure learning actions are intended to improve the information about (but not to directly affect) a quantity, potential sales price, etc. Exploratory drilling for example will improve information about the value of an oil field, and marketing research will help to better assess market share, etc. Learning actions are thus activated when a parameter significant for the decision making process is estimated with error. Management intends to eliminate or at least reduce this error in order to make optimal investment decisions.

If uncertainty is fully resolved, exercise of an investment option on stochastic asset S^* with exercise cost X yields $S^* - X$. Has a learning action not been taken before the investment decision is made, resolution of uncertainty (learning) would occur ex post. Ex ante, the investment decision must be made based solely on expected (instead of actual) outcomes. In this case exercise of the real option is expected to provide $E[S^*] - X$. The real investment prospect is a claim contingent on $S = E[S^*]$, and we assume that $E[S^*]$ follows a geometric Brownian motion, just like S^*. Thus, S will follow the same process before and after learning. Consider for example the case where the underlying asset is a product of two variables, a stochastic but observable one, and a constant but unobservable one. We seek to learn about the unobservable entity, and in doing so we will not affect the stochastic process of the product of the two. At learning, we will simply revise our estimate of the product (which will occur as a jump). *Fully revealing* learning actions are designed to resolve uncertainty completely (assuming this is feasible), but in the most general case *partly revealing* actions will be employed either because complete resolution of uncertainty is infeasible, or it is too costly. Each firm faces an investment decision, and either $S = E[S^*]$ is common for both (or simply differs by a constant), or each firm´s claim is contingent on a different asset, simply necessitating separate notation for S_1 and S_2 which again follow geometric Brownian motions.

We assume that each underlying asset (project) value, S, subject to i optional (and often costly) controls, follows a stochastic process of the form:

$$\frac{dS}{S} = \mu dt + \sigma dZ^R + \sum_{i=1}^{N} (k_i dq_i)\,, \tag{1}$$

where μ is the instantaneous expected return (drift) and σ the instantaneous standard deviation, dZ^R is an increment of a standard Wiener process in the real probability measure, and dq_i is a jump counter for managerial activation of action i -- a control (not a random) variable. Under risk-neutral valuation, the asset value S follows the process

$$\frac{dS}{S} = (r - \delta)dt + \sigma dZ + \sum_{i=1}^{N}(k_i dq_i) \qquad (1a)$$

where r is the riskless rate of interest, while the parameter δ represents any form of a "dividend yield" (e.g., in McDonald and Siegel, 1984, δ is a deviation from the equilibrium required rate of return, while in Brennan, 1991, δ is a convenience yield). As in Constantinides (1978) we need to assume that an intertemporal capital asset pricing model as in Merton (1973) holds. As in Merton (1976), we assume the jump (control) risk to be diversifiable (and hence not priced).

For each control i, we assume that the distribution of its size, $1 + k_i$, is log-normal, i.e., $\ln(1 + k_i) \sim \mathbf{N}(\gamma_i - .5\sigma_{C_i}^2,\ \sigma_{C_i}^2)$, with $\mathbf{N}(.,.)$ denoting the normal density function with mean $\gamma_i - .5\sigma_{C_i}^2$ and variance $\sigma_{C_i}^2$, and $E[k_i] \equiv \bar{k}_i = \exp(\gamma_i) - 1$. The control outcome is assumed independent of the Brownian motion -- although in a more general setting it can be dependent on time and/or the value of S. In general we can assume any plausible form, but the log-normality assumption will allow analytic tractability. Stochastic differential equation (1a) can alternatively be expressed in integral form as:

$$\ln[S(T)] - \ln[S(0)] = \int_0^T [r - \delta - .5\sigma^2]\, dt + \int_0^T \sigma dZ(t) + \sum_{i=1}^{N} dq_i \ln(1 + k_i)\,. \qquad (2)$$

Conditional on control activation

$$E[S^* \mid \text{activation of control } i] = E[S^*](1 + \bar{k}_i) = S(1 + \bar{k}_i)$$

and if the control is a pure learning (information acquisition) action ($\bar{k}_i = 0 = \gamma_i$)

$$E[S^* \mid \text{activation of control } i] = S\ .$$

Each firm's management seeks to optimally activate the control, so that each firm´s claim F on the underlying asset must satisfy (subject to the actions of the other firm) the following optimization problem:

$$Maximize[F(t, S, C)] \qquad (3)$$

subject to:

$$\frac{dS}{S} = (r - \delta)dt + \sigma dZ + \sum_{i=1}^{N} (k_i dq_i)$$

and

$\ln(1 + k_i)$ is normally distributed with mean: $\gamma_i - .5\,\sigma_{C_i}^2$, and variance: $\sigma_{C_i}^2$.

The next follows directly from the log-normality assumption of the multiplicative controls.

Proposition 1: *Assuming independence between the controls and the increment dZ of the standard Wiener process, the conditional solution to the European call option is given by:*

$$F(S, X, T, \sigma, \delta, r; \gamma_i, \sigma_{C_i}^2) = e^{-rT} E[(S^*_T - X,\ 0)^+ \mid \text{activation of the control}]\ . \qquad (4)$$

The present value of the risk-neutral expectation E[.] conditional on activation of the controls at t = 0, is isomorphic to the Black-Scholes (1973) model:

$E[(S^*_T - X, 0)^+ \mid$ activation of the controls at $t = 0] =$

$$S\ e^{(rT - \delta T + \sum_{i=1}^{N} f_i \gamma_i)} N(d_1) - X\ N(d_2) \tag{5}$$

where

$$d_1 \equiv \frac{\ln(S/X) + (r-\delta)T + \sum_{i=1}^{N} f_i \gamma_i + .5\sigma^2 T + .5\sum_{i=1}^{N} f_i \sigma_{C_i}^2}{[\sigma^2 T + \sum_{i=1}^{N} f_i \sigma_{C_i}^2]^{1/2}}$$

and

$$d_2 \equiv d_1 - [\sigma^2 T + \sum_{i=1}^{N} f_i \sigma_{C_i}^2]^{1/2}.$$

The $N(d)$ denotes the cumulative standard normal density evaluated at d. The degree of spillovers (parameter) f is shown more clearly later and may differ between firms. A control can be activated intentionally due to the manager´s learning activity, or unintentionally due to a spillover effect from the actions of the other firm´s management. For the first player, impact, learning, and spillover parameters with subscript one define results of own actions, and with subscript two the spillover effects. The opposite holds for the other firm. Given activation of learning controls, the (risk-neutral) probability $P(S_T > X)$ that the call option will be exercised at maturity T equals $N(d_2)$. At the moment of activation of the controls and due to the controls only, the probability that the new value of S will exceed some value X equals $N(d_2)$ with $T \to 0$, since the outcome of the controls is observed instantaneously. Finally the cost of controls must be subtracted from the *conditional* real option value for each firm. In pure learning actions intended just to resolve uncertainty about the true value of the unobservable variable S^* the impact parameters $\gamma_i = 0$. In the most general R&D case the impact parameters would differ from zero, and they can be positive if they affect own revenues or negative if they affect fixed or variable costs. The spillover impact to the other player can have either sign.

The Tactical Resource AllocationDecision.

The two managers must solve their optimization problem simultaneously, seeking thus an equilibrium in this (tactical) decision (see Figure 1). Let us denote with β_1 and β_2 the cost (effort) of the first and the second firm´s actions. The impact on option value is a function of that cost. Given the actions β_2 of the second firm, the impact on the net option value of firm one equals $F_1(\beta_1 \mid \beta_2) - \beta_1$, and given β_2 the first firm must maximize $F_1 - \beta_1$ through the first order condition

$$\frac{\partial[F_1(\beta_1 \mid \beta_2) - \beta_1]}{\partial \beta_1} = 0 \,. \qquad (6)$$

Similarly, firm two conditional on the first firm´s action β_1 must maximize $F_2(\beta_2 \mid \beta_1) - \beta_2$ through the first order condition

$$\frac{\partial[F_2(\beta_2 \mid \beta_1) - \beta_2]}{\partial \beta_2} = 0 \,. \qquad (6a)$$

The first order conditions are necessary for the existence of a maximum. Furthermore, if the second order conditions (that the second derivative is negative) are satisfied everywhere (or at least in some admissible range), this maximum, is unique (in the admissible range).

As shown in Figures 1(b) and 1(a), the optimal cost effort functions $\beta_1^*(\beta_2)$ and $\beta_2^*(\beta_1)$ for each firm are depended on the other firm’s actual effort. In this duopolistic game, both firms optimize their actions simultaneously and the equilibrium solution pair β_1^{**} and β_2^{**} is shown in Figure 1(c) at the intersection of $\beta_1^*(\beta_2)$ and $\beta_2^*(\beta_1)$. Since the cost efforts β_1 and β_2 affect F_1 and F_2 through the *impact* (γ_i) and *learning* (σ_{C_i}) parameters, we must define the mappings $\beta_1(\gamma_1, \sigma^2_{C_1})$ and $\beta_2(\gamma_2, \sigma^2_{C_2})$. As will be seen in the examples presented later, it is more intuitive to optimize directly with respect to the learning (or the impact) parameter. In order to solve numerically these two equations in the most general case, the 2x2 Jacobian matrix of the 2nd order analytic derivatives is needed and an iterative two-dimensional Newton-Raphson scheme is implemented.

Lemma 1: *The equilibrium efforts of the tactical decision are invariant to identically proportional changes in the price of the underlying asset S, the exercise price X, and the cost β of the control. The constant of proportionality may differ between the two players.*

Proof: It is easy to verify through the use of equations (6) – (6a) that the property of option prices to be homogenous of degree one in the underlying asset and the exercise price, is also preserved in this game theoretic context, due to the multiplicative nature of the random control. With the proper choice of the cost function, the conditional option prices (for each player) can be homogeneous of degree one in the underlying asset, the exercise price, and the control's cost θ_j, as clearly seen in equations (10a, b) and (12a, b).

The Strategic Coordination Decision.

We now must consider the strategic coordination choice. Firms must decide on the optimal degree of coordination of their R&D efforts. For example, in a 2x2 game, each firm can decide to exert high (*H*) or low (*L*) coordination effort (see Figure 2). The degree of coordination determines the extent of spillover effects (through the parameter *f*), and potentially the cost of R&D. We assume for ease of exposition two strategies available for each player, but more than two or even a continuous set of alternatives could exist. The choice sets (*H*, *H*), (*H*, *L*), etc. uniquely determine the degree of spillover effects. The optimal choice for the two firms is provided by the pure *Nash* equilibrium(a), or alternatively the mixed strategies equilibrium. Equilibria off the diagonal can occur because of the asymmetry in the direct spillover effects and the cost reduction results of coordination. This is justifiable when the two firms operate in different product markets, and can be for example technology dependent. The solution to the tactical decision in Figure 1 is nested to the solution of the strategic one in Figure 2.

Proposition 2: *If the constant of proportionality (as discussed in lemma* 1*) is the same for both players, then the Nash equilibrium strategy is invariant to the choice of this constant.*

Applications and numerical results: The pure learning case and the impact control case.

Let us first consider an example of pure learning actions. Two companies face an investment opportunity each. Before they decide to invest they also have the option to invest in order to acquire information (like in the case of oil exploration, marketing research, etc.) about the true value of the investment. Thus, both impact parameters equal zero since they do not pursue to directly enhance value but they do so indirectly by reducing uncertainty. We assume a cost function with the information revelation potential bounded from above (as well as positive). We consider *i* (= 1, 2) pure learning actions (γ_i = 0), one from each player *j* (= 1, 2), with *i* = *j* implying the player's action. To find the equilibrium, each player must maximize his/her option value:

$$F_j - \beta_j = S_j e^{(-\delta_j T)} N(d_1) - X_j e^{-rT} N(d_2) - \beta_j \tag{7}$$

where

$$d_1 \equiv \frac{\ln(S_j / X_j) + (r - \delta_j)T + 0.5\sigma_j^2 T + 0.5\sum_{i=1}^{2} f_{i \to j}\sigma_{C_i}^2}{[\sigma_j^2 T + \sum_{i=1}^{2} f_{i \to j}\sigma_{C_i}^2]^{1/2}}$$

and

$$d_2 \equiv d_1 - [\sigma_j^2 T + \sum_{i=1}^{2} f_{i \to j}\sigma_{C_i}^2]^{1/2}$$

given the action of the other player. The parameters $f_{i\to j}$ define the degree of spillovers. For the influence of own actions, most often $f_{1\to 1} = f_{2\to 2} = 1$. The cost is defined for simplicity linear in the variance of the learning action

$$\beta_j = \theta_j f_{j \to j}\sigma_{C_j}^2 . \tag{8}$$

The strategic decision determines the cost θ and the spillover f parameters.

Finally, the two equations

$$\frac{\partial(F_1 - \theta_1 f_{1\to 1}\sigma_{C_1}^2)}{\partial \sigma_{C_1}} = 0 , \tag{9a}$$

and

$$\frac{\partial(F_2 - \theta_2 f_{2\to 2}\sigma_{C_j}^2)}{\partial \sigma_{C_2}} = 0 \tag{9b}$$

are conditional on the other player's move, and must be solved simultaneously. We thus get,

$$\frac{\partial(F_1 - \theta_1 f_{1\to1}\sigma^2_{C_j})}{\partial\sigma_{C_1}} =$$

$$X_1 e^{-rT}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{f_{1\to1}\sigma_{C_1}}{[\sigma_1^2 T + f_{1\to1}\sigma^2_{C_1} + f_{2\to1}\sigma^2_{C_2}]^{0.5}} - 2\theta_1 f_{1\to1}\sigma_{C_1} = 0 \tag{10a}$$

and

$$\frac{\partial(F_2 - \theta_2 f_{2\to2}\sigma^2_{C_2})}{\partial\sigma_{C_2}} =$$

$$X_2 e^{-rT}\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{f_{2\to2}\sigma_{C_2}}{[\sigma_2^2 T + f_{2\to2}\sigma^2_{C_2} + f_{1\to2}\sigma^2_{C_1}]^{0.5}} - 2\theta_2 f_{2\to2}\sigma_{C_2} = 0 \tag{10b}$$

After solving for the optimal learning efforts

$$\sigma^{2**}_{C_1} \text{ and } \sigma^{2**}_{C_2},$$

we find the optimal cost efforts

$$\beta_1^{**} \text{ and } \beta_2^{**}.$$

In the application we assume that the optimal learning efforts are below the upper bound for the maximum feasible learning without loss of generality. Else, if the constraint were binding, in the iterative numerical solution we would have to incorporate it explicitly.

In the numerical example we assume that the spillovers are 50% (for $1 \to 2$) and 25% (for $2 \to 1$). Tables 1A and 1B show the learning effort and the cost involved for each player. As we see in Figure 3 equilibrium is when the first player spends 1.2047 and the second spends 3.0802. Their spending, results in information acquisition equivalent to σ_{C_1} = 0.10976 and σ_{C_2} = 0.20266, providing (through the use of equation 4) a net investment option value equal to 5.2756 and 5.5246 to player 1 and 2 respectively.

[Enter Tables 1A and 1B, and Figures 3, 3a, 3b, 3c about here]

As we can see from the following numerical applications, the general form of the response functions is as in Figure 3a with the unique solution at E. Figure 3b presents the case with the unique equilibrium E on the horizontal axis when player 2 gets a free lunch by exerting zero effort and benefiting from the spillovers from the actions of player 1 (symmetrically when player 1 gets the free lunch). These figures indicate the existence of a solution. The uniqueness of this solution depends on the slope and the curvature of the response functions. Figure 3c for example, presents a hypothetical case with three equilibria (E_1, E_2 and E_3). In the numerical examples presented in this paper, such a case has not been observed. Still, we cannot exclude the possibility of the existence of an infinite number of equilibria when the response functions coincide (such cases are observed and identified with asterisk in Tables 2 and 3).

In Table 2 we see the optimal decisions for a wide variety of asymmetric spillover effects and asymmetric costs. The degree of influence of these parameters on optimal effort is profound. When the underlying investment options and the R&D expenses are symmetric but the spillover effects are not, the player that receives less spillover benefits must spend more on R&D. When the spillover effects are symmetric but the R&D costs are not, the player that faces a steeper cost curve would rather reduce R&D spending, and oftentimes, we encounter *free-lunch* (almost zero effort on behalf of one player), as the preferred choice.

[Enter Tables 2, 3, and 4 about here]

While Table 2 investigates the case where the two firms face symmetric investment decisions, Table 3 investigates the case where the investment alternative of the first player is expected to be of larger scale than that of the second player, and Table 4 investigates the opposite case. As a result of Lemma 1, in the example discussed and for each player separately, we can multiply the price of the underlying asset S_1, the exercise price X_1, and the cost θ_1 with a positive constant, then multiply S_2, X_2, and θ_2 with another positive constant, and the results regarding the equilibrium effort will not change (compare for example the lower third of Table 2 with the middle third of Table 3). Optimal option values will of course change by the relevant constant.

Figure 4 presents the results from the strategic decision where the two players had to choose the optimal degree of R&D coordination. We see that both decide on the maximum degree of coordination. The first player however, decides not to spend on learning at all (see the results in parenthesis), whereas the second exhibits a very high effort. Figure 5 presents a case where the second player concedes to a high degree and the first to a low degree of coordination. Finally, figure 6 shows a case where the Nash equilibrium is a Low/Low strategy and figure 7 shows a case where the Nash equilibrium is a High/High strategy. Under no Nash equilibrium, mixed strategies could be considered as an alternative approach, providing the probabilities that each player would play a High or a Low strategy.

[Enter Figures 4, 5, 6 and 7 about here]

As a result of Proposition 2, if for the two players the constants of multiplication are the same, then the Nash equilibrium strategy of the games in figures 5-7 will also be unaffected, since all payoffs will be multiplied by the same positive constant.

The case of *impact* control.

In the previous application we focused on the pure learning (information acquisition) case. If the impact parameters are not zero (a direct effort to enhance value), this case of random control would be similarly solved through the use of the first order conditions (6) and (6a). Again we define each player's option value

$$F_j - \beta_j = S_j e^{(-\delta_j T + \sum_{i=1}^{2} f_{i\to j}\gamma_i)} N(d_1) - X_j e^{-rT} N(d_2) - \beta_j \tag{11}$$

where

$$d_1 \equiv \frac{\ln(S_j / X_j) + (r - \delta_j)T + \sum_{i=1}^{2} f_{i\to j}\gamma_i + 0.5\sigma_j^2 T + 0.5\sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2}{[\sigma_j^2 T + \sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2]^{1/2}},$$

$$d_2 \equiv d_1 - [\sigma_j^2 T + \sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2]^{1/2},$$

and for simplicity $\beta_j = \theta_j (f_{j\to j}\gamma_j)^2$ (or alternatively $\beta_j = \theta_j (f_{j\to j}\gamma_j) e^{(f_{j\to j}\gamma_j)}$, etc.)

For the tactical decision we consider the two first order conditions

$$\begin{aligned} &\frac{\partial [F_1 - \vartheta_1 (f_{1\to 1}\gamma_1)^2]}{\partial \gamma_1} = S_1 e^{(-\delta_j T + \sum_{i=1}^{2} f_{i\to 1}\gamma_i)} f_{1\to 1} N(d_1) \\ &+ X_1 e^{-rT} \frac{e^{-d_2^2/2}}{2\sqrt{2\pi}} \frac{s_{1\to 1} f_{1\to 1}\sigma_{C_1}^2}{[\sigma_1^2 T + s_{1\to 1} f_{1\to 1}\gamma_1\sigma_{C_1}^2 + s_{2\to 1} f_{2\to 1}\gamma_2\sigma_{C_2}^2]^{0.5}} - 2\theta_1 f_{1\to 1}^2\gamma_1 = 0 \end{aligned} \tag{12a}$$

and

$$\begin{aligned} &\frac{\partial [F_2 - \vartheta_2 (f_{2\to 2}\gamma_2)^2]}{\partial \gamma_2} = S_2 e^{(-\delta_j T + \sum_{i=1}^{2} f_{i\to 2}\gamma_i)} f_{2\to 2} N(d_1) \\ &+ X_2 e^{-rT} \frac{e^{-d_2^2/2}}{2\sqrt{2\pi}} \frac{s_{2\to 2} f_{2\to 2}\sigma_{C_2}^2}{[\sigma_2^2 T + s_{1\to 2} f_{1\to 2}\gamma_1\sigma_{C_1}^2 + s_{2\to 2} f_{2\to 2}\gamma_2\sigma_{C_2}^2]^{0.5}} - 2\theta_2 f_{2\to 2}^2\gamma_2 = 0 \end{aligned} \tag{12b}$$

where the constants $s_{i\to j}$ simply guarantee the positivity of the variance components. Subsequently and similarly with the pure learning case we solve the two equations simultaneously and we get the optimal impact efforts γ_1^{**} and γ_2^{**} and through them the optimal cost efforts β_1^{**} and β_2^{**}. Again we can assume that the impact parameters are bounded from above. An example for the cost function $\beta_j = \theta_j (f_{j\to j}\gamma_j)^2$, with the range of admissible impact parameters bounded below 100% is given in Figure 8 and Tables 4A and 4B (for the solution to the tactical decision). Figures 9 and 10 present two examples where the Nash equilibrium for the strategic decision is the highest (H/H) and the lowest (L/L) degree of coordination respectively.

[Enter figures 8, 9, 10 and Tables 4A and 4B about here]

In Figure 9 the impact spillover is positive (like generic advertisement) whereas in Figure 10 the impact spillover can be negative when advertisement is more competitive and less costly.

Conclusions

This paper presents and solves a real options duopoly game that jointly addresses at the pre-investment stage the strategic decision about the extent of coordination between two firms, and the tactical decision about the optimal effort invested in R&D in the presence of uncertainty and spillover effects. We assume that the two firms can influence each other´s decision at the pre-investment stage, whereas at the investment decision each firm has monopoly power over its investment and there is no further interaction between the two. Managers want to enhance value and to resolve (or reduce) uncertainty of real (investment) opportunities, before they make a commitment. Learning actions are treated as controlled jumps of random size whose realization is a random variable with a known probability distribution. We used a contingent claims framework with incomplete information and costly control actions, and without use of generality or any sacrifice in insights gained we made the assumption that the two firms face investment opportunities of the European type, allowing thus the use of analytic models isomorphic to Black and Scholes (1973). Alternatively, fully numerical methods like lattice or numerical solutions to partial differential equations could have been used, but the iterative solution to this continuous game would have been much more intensive computationally and less accurate.

Finally, the solution to the firms' optimal strategic and tactical R&D decision-making is found as the equilibrium of a two-stage game. This decision, as expected, is heavily dependent on the learning potential of R&D investments, their cost, and the degree of coordination that is optimal for the two players. Some times *high* coordination and other times *low* coordination will be optimal for each player. The degree of coordination affects both the degree of spillovers, and the parameters of the cost function. There are cases where a player will *delegate* research by agreeing on a high degree of coordination (lowering thus the R&D cost of the other player) and reap through the spillovers the rewards. In general we can say that, as we have shown, optimal coordination and optimal investment in information acquisition and value enhancement are essential for optimal investment decision-making.

References

Bernstein, J. I., and M. I. Nadiri, 1988. Interindustry R&D Spillovers, Rates of Return, and production in High-Tech Industries. *American Economic Review* 78, 429-434.

Black, F., and M. Scholes, 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81, 637-659.

Branstetter, L., 1996. Are knowledge spillovers international or intranational in scope? Microeconomic evidence from the US and Japan. National Bureau of Economic Research Working Paper: 5800.

Brennan, M.J., 1991. The price of convenience and the valuation of commodity contingent claims. In Lund D., and Øksendal, B. (Eds.), *Stochastic Models and Option Values*. North-Holland, 33-72.

Brod, A., and R. Shivakumar, 1999. Advantageous semi-collusion. *Journal of Industrial Economics* XLVII, 221-230.

Carey, C., and D. L. Bolton, 1996. Brand versus generic advertising and the decision to advertise collectively. *Review of Industrial Organization* 11, 93-105.

Childs, P. D., S. H. Ott, and T. J. Riddiough, 1999. Valuation and information acquisition policy for claims written on noisy real assets. Working Paper presented at the 3rd Annual International Conference on Real Options, NIAS – Leiden, The Netherlands.

Constantinides, G., 1978. Market risk adjustment in project valuation. *Journal of Finance* 33, 603-616.

d´Aspremont, C., and A. Jacquemin, 1988. Cooperative and noncooperative R&D in duopoly with spillovers. *American Economic Review* 78, 1133-1137.

Dixit, A.K., and R.S. Pindyck, 1994. *Investment Under Uncertainty*. Princeton University Press, Princeton: New Jersey.

Epstein, D., N. Mayor, P. Schonbucher, E. Whaley, and P. Wilmott, 1999. The value of market research when a firm is learning: Real option pricing and optimal filtering. In *Real options and business strategy. Applications to decision making*, L. Trigeorgis (ed.). Risk Books, London, UK.

Fershtman, C., and N. Gandal, 1994. Disadvantageous semicollusion. *International Journal of Industrial Organization* 12, 141-154.

Foster, A. D., and M. R. Rosenzweig, 1995. Learning by doing and learning from others: Human capital and technical change in agriculture. *Journal of Political Economy* 103, 1176-1209.

Grenadier, S. R., 1996. The strategic exercise of options: Development cascades and overbuilding in real estate markets. *Journal of Finance* 51, 1653-1679.

Jaffe, A. B., and M. Trajtenberg, 1998. International knowledge flows: evidence from patent citations. National Bureau of Economic Research Working Paper: 6507.

Johnson, D. K. N., and R. E. Evenson, 1999. R&D spillovers to agriculture: Measurement and application. *Contemporary Economic Policy* 17, 432-456.

Kamien, M. I., E. Muller, and I. Zang, 1992. Research joint ventures and R&D cartels. *American Economic Review* 82, 1293-1306.

Majd, S., and R. Pindyck, 1989. The learning curve and optimal production under uncertainty. *RAND Journal of Economics* 20, 331-343.

Martzoukos, S.H., 2000, Real options and the value of learning. *Annals of Operations Research* 99, 305-323.

Martzoukos, S.H., and L. Trigeorgis, 2000. Resolving a real options paradox with incomplete information: After all, *why learn*? Working paper, Department of Public and Business Administration, University of Cyprus. Presented at the 5th Annual International Conference on Real Options, UCLA (July 2001).

McDonald, R., and D. Siegel, 1984. Option pricing when the underlying asset earns a below-equilibrium rate of return: A note. *Journal of Finance* 39, 261-265.

Merton, R. C., 1973. An intertemporal capital asset pricing model. *Econometrica* 41, 867-887.

Merton, R. C., 1976. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125-144.

Paddock, J. L., D. Siegel, and J.L. Smith, 1988. Option valuation of claims on real assets: The case of offshore petroleum leases. *Quarterly Journal of Economics* 103, 479-508.

Pennings, E., and Lint, O., 1997. The option value of advanced R&D. *European Journal of Operational Research* 103, 83-94.

Poyago-Theotoky, J., 1999. A Note on Endogenous Spillovers in a Non-Tournament R&D Duopoly. *Review of Industrial Organization* 15, 253-262.

Roberts, K., and M. Weitzman, 1981. Funding criteria for research, development and exploration projects. *Econometrica* 49, 1261-1288.

Smit, H. T. J., and L. A. Ankum, 1993. A real options and game-theoretic approach to corporate investment strategy under competition. *Financial Management* 22, 241-250.

Sundaresan, S. M., 2000. Continuous-time methods in finance: A review and an assessment. *The Journal of Finance* 55, 1569-1622.

Tao, Z., and C. Wu, 1997. On the organization of cooperative research and development: theory and evidence. *International Journal of Industrial Organization* 15, 573-596.

Thompson, P., 2000. Learning from the experience of others: Parameter uncertainty and economic growth in a model of creative destruction. *Journal of Economic Dynamics and Control* 24, 1285-1314.

Trigeorgis, L., 1996. *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. The MIT Press.

Williams, J., 1993. Equilibrium and options on real assets. *Review of Financial Studies* 6, 825-850.

Table 1A

Pure learning response functions

Player 1 optimizing (given 2's effort)

Total learning effort		Learning cost		Net call option value	
1's effort	2's effort	1's cost	2's cost	1's option	2's option
0.14028	0.10266	1.9680	0.7904	4.5123	5.4933
0.13506	0.12766	1.8240	1.2222	4.6563	5.5611
0.12841	0.15266	1.6488	1.7478	4.8314	5.5974
0.12010	0.17766	1.4424	2.3671	5.0379	5.5888
0.10976	0.20266	1.2047	3.0802	5.2756	5.5246
0.09673	0.22766	0.9357	3.8870	5.5445	5.3964
0.07972	0.25266	0.6355	4.7876	5.8447	5.1979
0.05515	0.27766	0.3041	5.7819	6.1762	4.9241
0.00000	0.30266	0.0000	6.8700	6.5386	4.5876

Table 1B

Pure learning response functions

Player 2 optimizing (given 1's effort)

Total learning effort		Learning cost		Net call option value	
1's effort	2's effort	1's cost	2's cost	1's option	2's option
0.00976	0.21690	0.0095	3.5284	5.3223	5.0764
0.03476	0.21561	0.1208	3.4866	5.3283	5.1181
0.05976	0.21285	0.3571	3.3980	5.3329	5.2067
0.08476	0.20857	0.7184	3.2625	5.3212	5.3422
0.10976	0.20266	1.2047	3.0802	5.2756	5.5246
0.13476	0.19497	1.8160	2.8509	5.1785	5.7538
0.15976	0.18529	2.5523	2.5748	5.0148	6.0299
0.18476	0.17328	3.4136	2.2518	4.7720	6.3529
0.20976	0.15841	4.3998	1.8820	4.4401	6.7228

Notes (for 1A and 1B). Investment options' parameters are: underlying assets $S_1 = S_2 = 100.00$, exercise prices $X_1 = X_2 = 100.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$, and costs of learning (per unit of variance) $\theta_1 = 100.00$ and $\theta_2 = 75.00$ with spillover of learning 50% (for $1 \to 2$) and 25% (for $2 \to 1$).

Table 2
Optimal R&D learning effort (player 1 / player 2)

Cost θ_2	Spillovers $2 \rightarrow 1$	Spillovers $1 \rightarrow 2$				
		0.00	0.25	0.50	0.75	1.00
50.00	0.00	0.14938 0.34096	0.14938 0.33268	0.14938 0.32419	0.14938 0.31547	0.14938 0.30650
	0.25	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	0.50	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	0.75	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	1.00	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
75.00	0.00	0.14938 0.21701	0.14938 0.20375	0.14938 0.18957	0.14938 0.17423	0.14938 0.15741
	0.25	0.10267 0.21701	0.10604 0.21043	0.10976 0.20266	0.11390 0.19329	0.11855 0.18176
	0.50	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
	0.75	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
	1.00	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
100.00	0.00	0.14938 0.14938	0.14938 0.12937	0.14938 0.10563	0.14938 0.07469	0.14938 0.00000
	0.25	0.12937 0.14938	0.13361 0.13361	0.13830 0.11292	0.14352 0.08286	0.14938 0.00000
	0.50	0.10563 0.14938	0.11292 0.13830	0.12197 0.12197	0.13361 0.09448	0.14938 0.00000
	0.75	0.07469 0.14938	0.08286 0.14352	0.09448 0.13361	0.11292 0.11292	0.14938 0.00000
	1.00	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	*0.10563 0.10563

Notes. First number for player 1 and second for player 2. $S_1 = X_1 = 100.00$ and $S_2 = X_2 = 100.00$. Other real investment option parameters as in Tables 1A and 1B.

* Due to the symmetry of the assumptions for the two players, the response functions coincide and this point is approximated at the limit from $f_{1\rightarrow 2} = f_{2\rightarrow 1} \rightarrow 1^-$ or $f_{1\rightarrow 2} = f_{2\rightarrow 1} \rightarrow 1^+$.

Table 3
Optimal R&D learning effort (player 1 / player 2)

Cost θ_2	Spillovers $2 \to 1$	Spillovers $1 \to 2$				
		0.00	0.25	0.50	0.75	1.00
50.00	0.00	0.14938 0.24898	0.14938 0.23751	0.14938 0.22546	0.14938 0.21273	0.14938 0.19919
	0.25	0.08256 0.24898	0.08527 0.24530	0.08826 0.24103	0.09160 0.23600	0.09534 0.23000
	0.50	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
	0.75	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
	1.00	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898	0.00000 0.24898
75.00	0.00	0.14938 0.14938	0.14938 0.12937	0.14938 0.10563	0.14938 0.07469	0.14938 0.00000
	0.25	0.12937 0.14938	0.13361 0.13361	0.13830 0.11292	0.14352 0.08286	0.14938 0.00000
	0.50	0.10563 0.14938	0.11292 0.13830	0.12197 0.12197	0.13361 0.09448	0.14938 0.00000
	0.75	0.07469 0.14938	0.08286 0.14352	0.09448 0.13361	0.11292 0.11292	0.14938 0.00000
	1.00	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	*0.10563 0.10563
100.00	0.00	0.14938 0.09078	0.14938 0.05160	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.25	0.14232 0.09078	0.14698 0.05329	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.50	0.13488 0.09078	0.14420 0.05516	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	0.75	0.12702 0.09078	0.14091 0.05724	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000
	1.00	0.11863 0.09078	0.13698 0.05958	0.14938 0.00000	0.14938 0.00000	0.14938 0.00000

Notes. First number for player 1 and second for player 2. $S_1 = X_1 = 100.00$ and $S_2 = X_2 = 75.00$. Other real investment option parameters as in Tables 1A and 1B.
* Due to the relative symmetry of the assumptions for the two players, the response functions coincide and this point is approximated at the limit from $f_{1\to2} = f_{2\to1} \to 1^-$ or $f_{1\to2} = f_{2\to1} \to 1^+$.

Table 4
Optimal R&D learning effort (player 1 / player 2)

Cost θ_2	Spillovers $2 \to 1$	Spillovers $1 \to 2$				
		0.00	0.25	0.50	0.75	1.00
50.00	0.00	0.09078 0.34096	0.09078 0.33793	0.09078 0.33487	0.09078 0.33178	0.09078 0.32866
	0.25	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	0.50	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	0.75	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
	1.00	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096	0.00000 0.34096
75.00	0.00	0.09078 0.21701	0.09078 0.21221	0.09078 0.20730	0.09078 0.20227	0.09078 0.19711
	0.25	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
	0.50	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
	0.75	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
	1.00	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701	0.00000 0.21701
100.00	0.00	0.09078 0.14938	0.09078 0.14232	0.09078 0.13488	0.09078 0.12702	0.09078 0.11863
	0.25	0.05160 0.14938	0.05329 0.14698	0.05516 0.14420	0.05724 0.14091	0.05958 0.13698
	0.50	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938
	0.75	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938
	1.00	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938	0.00000 0.14938

Notes. First number for player 1 and second for player 2. $S_1 = X_1 = 75.00$ and $S_2 = X_2 = 100.00$. Other real investment option parameters as in Tables 1A and 1B.

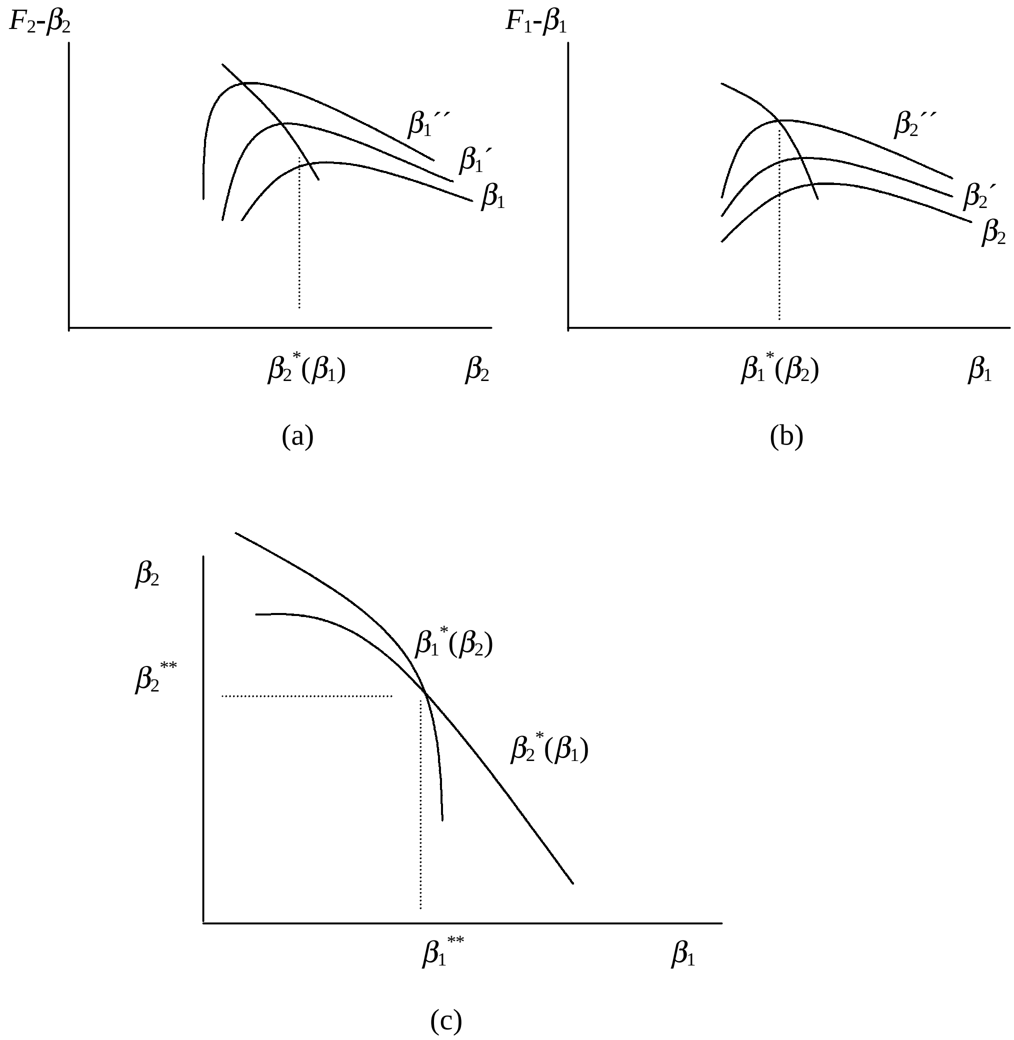

Figure 1
Game with Spillovers: Tactical Resource Allocation Decision

Notes: β_1, β_2 are the learning costs incurred by the two firms, $F_1(\beta_1 \mid \beta_2)$, $F_2(\beta_2 \mid \beta_1)$ are the investment option values before learning costs are subtracted, and $\beta_1^*(\beta_2)$, $\beta_2^*(\beta_1)$ are the optimal cost efforts of each player (*conditional* on the effort of the other). The equilibrium solution pair (β_1^{**}, β_2^{**}) is given by the intersection of the two optimal *conditional* cost effort curves.

		FIRM 1: *H*	*FIRM* 1: *L*
FIRM 2	*H*	$F_1(H, H)$, $F_2(H, H)$	$F_1(L, H)$, $F_2(L, H)$
	L	$F_1(H, L)$, $F_2(H, L)$	$F_1(L, L)$, $F_2(L, L)$

Figure 2
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) of Low (*L*) degree of R&D coordination. F_1 and F_2 are the investment option values (before the cost of investments in coordinated R&D are subtracted). $F_1(H, L) = F_1[\beta_1^{**}(H, L), \beta_2^{**}(H, L)]$, $F_2(H, L) = F_2[\beta_1^{**}(H, L), \beta_2^{**}(H, L)]$.

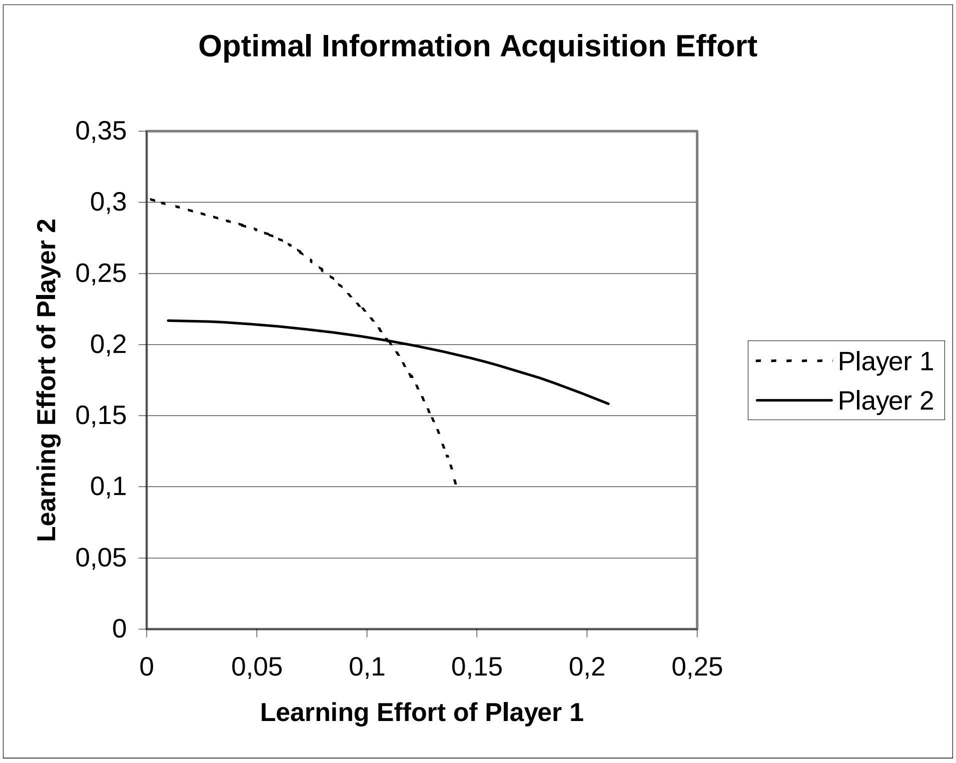
Optimal Information Acquisition Effort
0,35
0,3
0,25
0,2
0,15
0,1
0,05
0
Learning Effort of Player 2
0
0,05
0,1
0,15
0,2
0,25
Learning Effort of Player 1
Player 1
Player 2

Figure 3

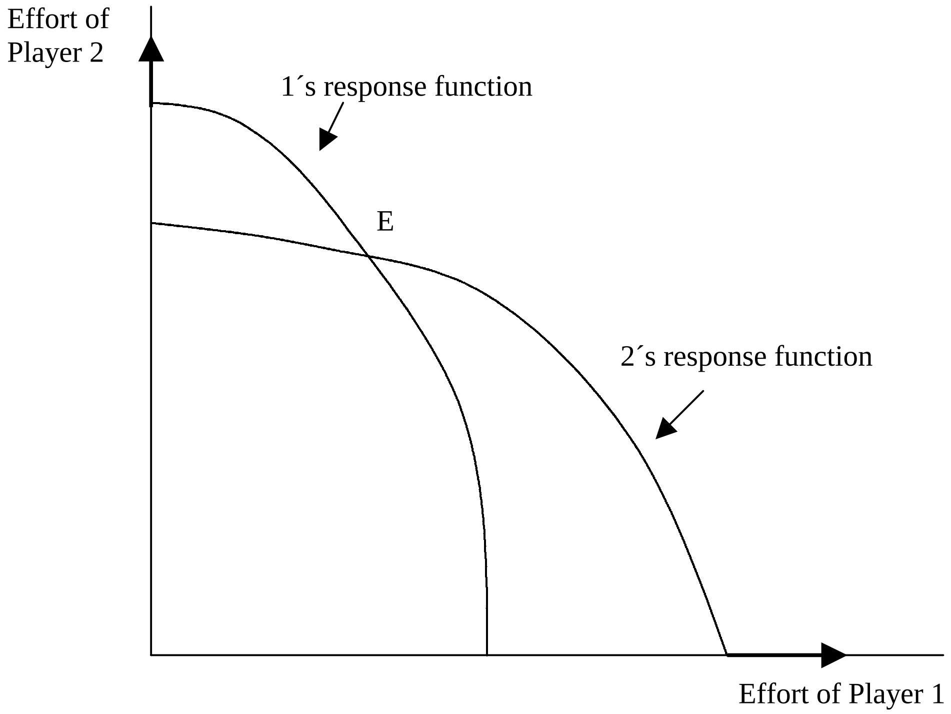

Figure 3a
Unique Optimal (Learning) R&D Solution – the general case

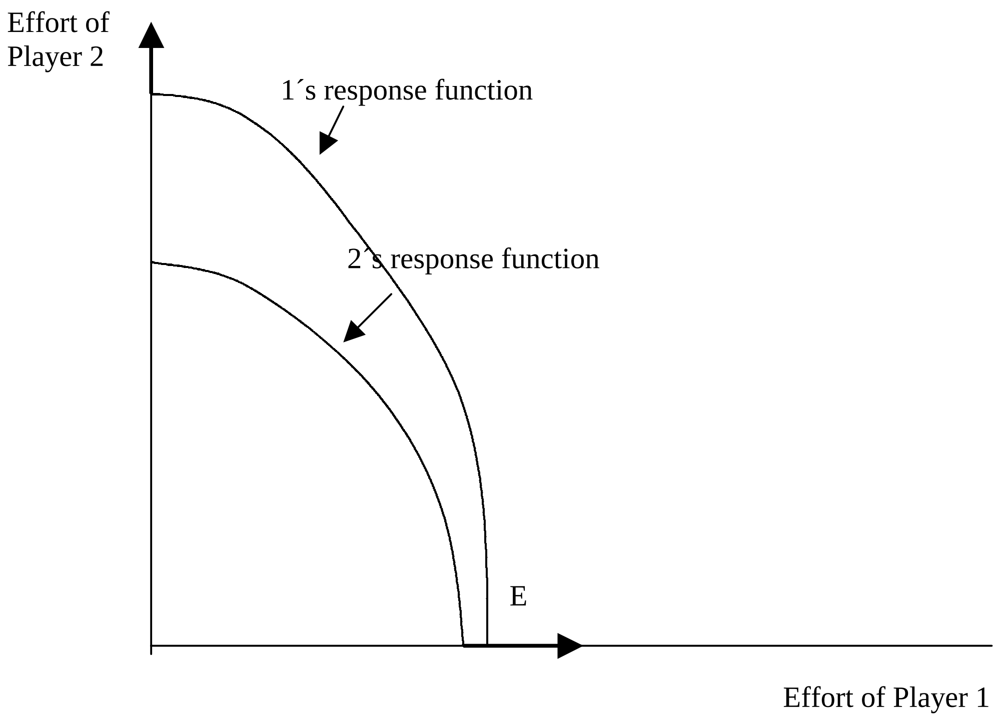

Figure 3b
Unique Optimal (Learning) R&D Solution – free lunch for player 2

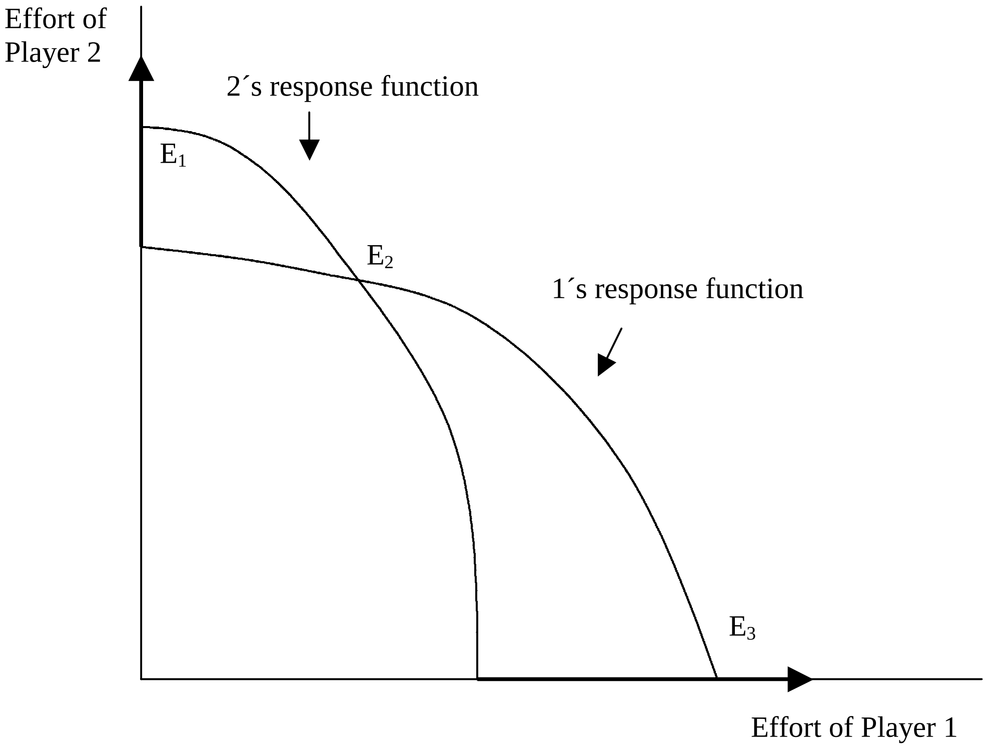

Figure 3c
Multiple Optimal (Learning) R&D Solutions

		FIRM 1	
		H	*L*
FIRM 2	*H*	**16.3155, 7.4114** (0.0000, 0.5136) 80.00, 25.00 0.75, 0.75	5.7594, 3.7296 (0.1702, 0.1228) 80.00, 75.00 0.75, 0.25
	L	9.9166, 7.4114 (0.0000, 0.5136) 100.00, 25.00 0.25, 0.75	4.5915, 3.4628 (0.1374, 0.1171) 100.00, 80.00 0.25, 0.25

Figure 4
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) of Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the players' effort. Below we provide the costs θ_1 and θ_2, and each player's degree of spillover from the other player's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

		FIRM 1	
		H	*L*
FIRM 2	*H*	4.5076, 4.8640 (0.1727, 0.0000) 90.00, 75.00 0.75, 0.75	**5.1513, 3.4829** (0.1506, 0.0977) 90.00, 85.00 0.75, 0.25
	L	4.4205, 4.3453 (0.1435, 0.0829) 100.00, 75.00 0.25, 0.75	4.5055, 3.4209 (0.1405, 0.1013) 100.00, 85.00 0.25, 0.25

Figure 5
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) of Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the players' effort. Below we provide the costs θ_1 and θ_2, and each player's degree of spillover from the other player's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

		FIRM 1	
		H	*L*
FIRM 2	*H*	4.5186, 3.8350 (0.1401, 0.0734) 100.00, 85.00 0.50, 0.50	4.7988, 3.3586 (0.1297, 0.1049) 100.00, 85.00 0.50, 0.25
	L	4.3645, 3.9005 (0.1455, 0.0680) 100.00, 85.00 0.25, 0.50	**4.5055, 3.4209** (0.1405, 0.1013) 100.00, 85.00 0.25, 0.25

Figure 6

Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) of Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the players' effort. Below we provide the costs θ_1 and θ_2, and each player's degree of spillover from the other player's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

		FIRM 1	
		H	*L*
FIRM 2	*H*	**6.2356, 4.3422** (0.1417, 0.1886) 82.00, 57.00 0.50, 0.50	5.5622, 3.7309 (0.1552, 0.1450) 85.00, 70.00 0.50, 0.25
	L	5.1596, 4.3185 (0.1502, 0.1702) 90.00, 60.00 0.25, 0.50	4.9779, 3.7400 (0.1568, 0.1446) 90.00, 70.00 0.25, 0.25

Figure 7
Game with Learning Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) of Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the players' effort. Below we provide the costs θ_1 and θ_2, and each player's degree of spillover from the other player's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

Table 4A
Impact response functions
Player 1 optimizing (given 2's effort)

Total impact effort		Impact cost		Net call option value	
1's effort	2's effort	1's cost	2's cost	1's option	2's option
0.08266	0.01785	3.4168	0.1594	6.2138	6.9726
0.08719	0.03785	3.8013	0.7165	7.0636	8.0927
0.09119	0.05785	4.1579	1.6736	7.9559	8.9596
0.09472	0.07785	4.4863	3.0307	8.8858	9.5515
0.09785	0.09785	4.7878	4.7878	9.8490	9.8491
0.10065	0.11785	5.0649	6.9449	10.8418	9.8354
0.10315	0.13785	5.3204	9.5020	11.8610	9.4971
0.10543	0.15785	5.5573	12.4591	12.9041	8.8243
0.10750	0.17785	5.7787	15.8162	13.9689	7.8101

Table 4B
Impact response functions
Player 2 optimizing (given 1's effort)

Total impact effort		Impact cost		Net call option value	
1's effort	2's effort	1's cost	2's cost	1's option	2's option
0.017855	0.082665	0.1594	3.4168	6.9726	6.2138
0.037855	0.087192	0.7165	3.8013	8.0927	7.0636
0.057855	0.091191	1.6736	4.1579	8.9596	7.9559
0.077855	0.094724	3.0307	4.4863	9.5515	8.8858
0.097855	0.097855	4.7878	4.7878	9.8491	9.8490
0.117855	0.100647	6.9449	5.0649	9.8354	10.8418
0.137855	0.103154	9.5020	5.3204	9.4971	11.8610
0.157855	0.105426	12.4591	5.5573	8.8243	12.9041
0.177855	0.107505	15.8162	5.7787	7.8101	13.9689

Notes (for 4A and 4B). Investment options' parameters are: underlying assets $S_1 = S_2 = 100.00$, exercise prices $X_1 = X_2 = 100.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$.

Costs of impact (per unit of squared impact) $\theta_1 = 500.00$ and $\theta_2 = 500.00$ with spillover of impact 50% (for $1 \to 2$) and 50% (for $2 \to 1$). Impact induced volatility equals $\sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2 = \sum_{i=1}^{2} | f_{i\to j}\gamma_i | 0.05^2$ for both players. Admissible impact range is bounded below 100%.

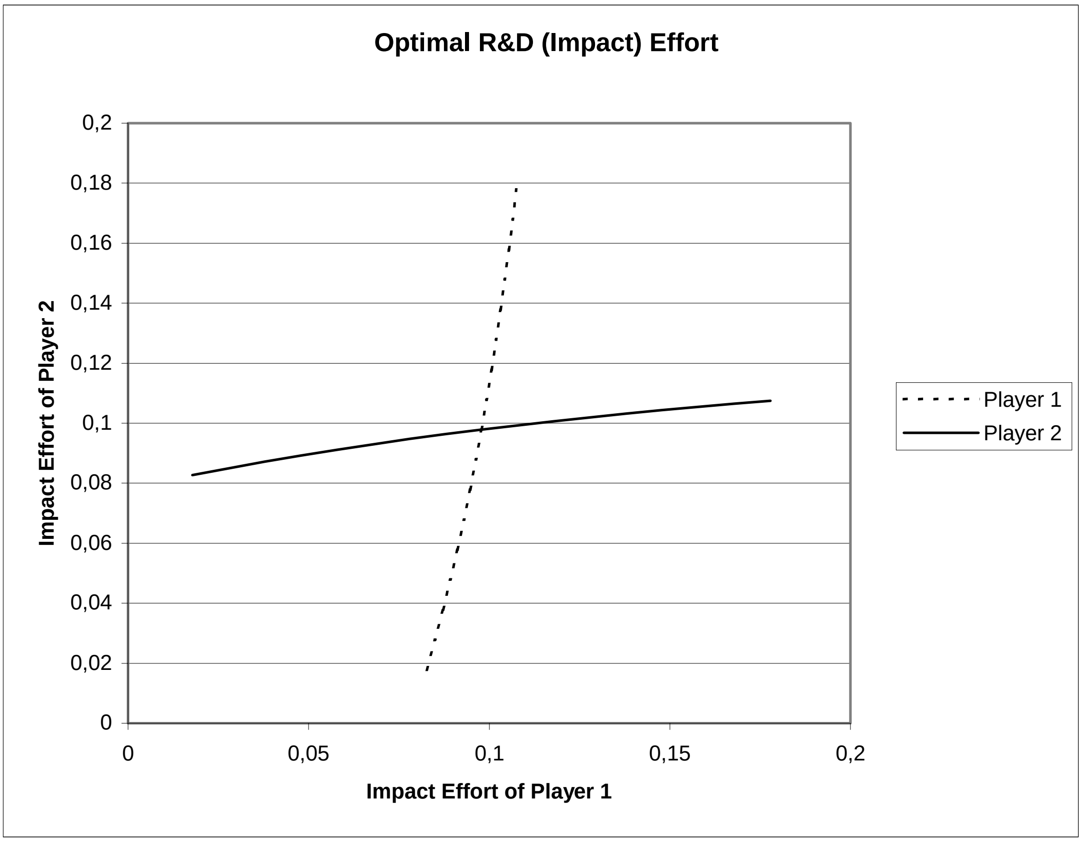
Optimal R&D (Impact) Effort
0,2
0,18
0,16
0,14
0,12
0,1
0,08
0,06
0,04
0,02
0
Impact Effort of Player 2
0
0,05
0,1
0,15
0,2
Impact Effort of Player 1
Player 1
Player 2

Figure 8

FIRM 1

FIRM 2	*H*	*L*
H	**21.3807, 16.4136** (0.2566, 0.1856) 250.00, 250.00 0.50, 0.50	14.2699, 7.2619 (0.2366, 0.0702) 250.00, 500.00 0.50, 0.25
L	9.1118, 9.3363 (0.0955, 0.1652) 500.00, 250.00 0.25, 0.50	6.7082, 4.8011 (0.0854, 0.0592) 500.00, 500.00 0.25, 0.25

Figure 9
Game with Impact Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) of Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the players' effort. Below we provide the costs θ_1 and θ_2, and each player's degree of spillover from the other player's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$. The impact induced volatility equals for both players $\sum_{i=1}^{2} | f_{i\to j}\gamma_i | \sigma_{C_i}^2 = \sum_{i=1}^{2} | f_{i\to j}\gamma_i | 0.05^2$. Admissible impact range is bounded below 100%.

		FIRM 1	
		H	*L*
FIRM 2	*H*	7.1494, 4.5636 (0.1153, 0.0578) 400.00, 500.00 0.15, 0.15	10.6827, 1.4637 (0.2251, 0.0271) 250.00, 500.00 0.15, -0.25
	L	3.0131, 6.1968 (0.0542, 0.1508) 500.00, 250.00 -0.25, 0.15	**9.1651, 1.6091** (0.2192, 0.0384) 250.00, 400.00 -0.25, -0.25

Figure 10
Game with Impact Spillovers: Strategic Coordination Decision

Notes: Firms 1 and 2 exhibit a High (*H*) of Low (*L*) degree of R&D coordination. The numerical results for the investment option values under optimal effort are presented first (with the Nash equilibrium in bold). Numbers in parenthesis present the players' effort. Below we provide the costs θ_1 and θ_2, and each player's degree of spillover from the other player's effort. Option parameter values are: $S_1 = X_1 = 100.00$, $S_2 = X_2 = 75.00$, dividend yields $\delta_1 = \delta_2 = 0.10$, riskless rate $r = 0.10$, standard deviation of the continuous change of the underlying assets $\sigma_1 = \sigma_2 = 0.10$, time to maturity for both options $T = 1.00$. The impact induced volatility equals for both players $\sum_{i=1}^{2}|f_{i\to j}\gamma_i|\sigma_{C_i}^2 = \sum_{i=1}^{2}|f_{i\to j}\gamma_i|0.05^2$. Admissible impact range is bounded below 100%.

WORKING PAPER SERIES

HERMES Center of Excellence on Computational Finance & Economics University of Cyprus

Preprints of these working papers can be downloaded from www.hermes.ucy.ac.cy or obtained by sending an e-mail to hermes@ucy.ac.cy

Working Paper 01-01	*STOCHASTIC PROGRAMMING MODELS FOR ASSET AND LIABILITY MANAGEMENT* *Roy Kouwenberg and Stavros A. Zenios*
Working Paper 01-02	*THE VALUE OF INTEGRATIVE RISK MANAGEMENT FOR INSURANCE PRODUCTS WITH GUARANTEES* *Andrea Consiglio, Flavio Cocco and Stavros A. Zenios*
Working Paper 01-03	*EXTENDING CREDIT RISK (PRICING) MODELS FOR THE SIMULATION OF PORTFOLIOS OF INTEREST RATE AND CREDIT RISK SENSITIVE SECURITIES* *Norbert Jobst and Stavros A. Zenios*
Working Paper 01-04	*TRACKING CORPORATE BOND INDICES IN AN INTEGRATED MARKET AND CREDIT RISK ENVIRONMENT* *Norbert Jobst and Stavros A. Zenios*
Working Paper 01-05	*THE TAIL THAT WAGS THE DOG: INTEGRATING CREDIT RISK IN ASSET PORTFOLIOS* *Norbert Jobst and Stavros A. Zenios*
Working Paper 01-06	*PARALLELIZATION, OPTIMIZATION, AND PERFORMANCE ANALYSIS OF PORTFOLIO CHOICE MODELS* *Ahmed Abdelkhalek, Angelos Bilas, and Alexander Michaelides*
Working Paper 01-07	*REAL R&D OPTIONS WITH ENDOGENOUS AND EXOGENOUS LEARNING* *Spiros H. Martzoukos*
Working Paper 01-08	*PORTFOLIO CHOICE AND LIQUIDITY CONSTRAINTS* *Michael Haliassos and Alexander Michaelides*

(Also published as CEPR Discussion Paper No. 2822)

Working Paper 01-09	*PORTFOLIO CHOICE, LIQUIDITY CONSTRAINTS AND STOCK MARKET MEAN REVERSION* *Alexander Michaelides*

(Also published as CEPR Discussion Paper No. 2823)

Working Paper 01-23 — *CVaR MODELS WITH SELECTIVE HEDGING FOR INTERNATIONAL ASSET ALLOCATION*
Nikolas Topaloglou, Hercules Vladimirou, Stavros A. Zenios

Working Paper 02-01 — *THE PROMETEIA MODEL FOR MANAGING INSURANCE POLICIES WITH GUARANTEES*
Andrea Consiglio, Flavio Cocco and Stavros A. Zenios

Working Paper 02-02 — *STOCHASTIC PROGRAMMING MODELS FOR MANAGING INTERNATIONAL INVESTMENT PORTFOLIOS*
Nikolas Topaloglou, Hercules Vladimirou, Stavros A. Zenios

Working Paper 02-03 — *LIFE-CYCLE ASSET ALLOCATION: A MODEL WITH BORROWING CONSTRAINTS, UNINSURABLE LABOR INCOME RISK AND STOCK-MARKET PARTICIPATION COSTS*
Francisco Gomes and Alexander Michaelides

Working Paper 02-04 — *CRITICAL ASSESSMENT OF OPTION PRICING METHODS USING ARTIFICIAL NEURAL NETWORKS AND IMPLIED VOLATILITY*
Panayiotis Ch. Andreou, Chris Charalambous and Spiros H. Martzoukos