REAL OPTIONS WITH INCOMPLETE INFORMATION AND MULTI-DIMENSIONAL RANDOM CONTROLS

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Real Options with Incomplete Information and Multi-Dimensional Random Controls

Abstract

In this paper we provide a real (investment) options´ valuation method with controls that capture managerial intervention and learning (exploration, R&D, advertising, marketing research, etc.). In contrast to the standard *wait-and-see* approach of the real options literature, we assume that managers possess the ability to intervene either for value enhancement, or for information acquisition, and of course they wish to do so optimally. We assume the presence of multiple stochastic state-variables that follow Geometric Brownian motion (GBM), or jump-diffusion processes. Activated controls can affect several or all of the state-variables, and the outcome of control activation is random. For the case first of GBM processes, an analytic solution is provided to value the real claim in the presence of such multi-dimensional controls, and a Markov-chain numerical method is demonstrated for more complex applications. The method of random controls is similarly extended to the case where the state-variables follow jump-diffusion processes, with multiple classes of jumps. An analytic solution is provided, and again a Markov-chain numerical method is demonstrated. Neglecting the effect of such actions, causes a serious underestimate of the value of real options and leads to erroneous decision making.

I. Introduction

Contingent claims valuation (see Black and Scholes, 1973, and Merton, 1973) has been extended to the case of valuation of investments under uncertainty (i.e., McDonald and Siegel, 1986; for extensive review see Dixit and Pindyck, 1994, and Trigeorgis, 1996). Numerical methods (for example the lattice techniques in Boyle, Evnine, and Gibbs, 1989, Kamrad and Ritchken, 1991, and Ekvall, 1996) allow valuation in the presence of several state-variables that follow GBM, for both European and American options, in simple and complex applications. The standard GBM assumption was extended in order to capture rare events by Merton (1976) to that of mixtures of GBM with randomly (Poisson) arriving rare events often described as jump-diffusion (see also Jones, 1984, Ball and Torous, 1985, Bates, 1991, and Amin, 1993).

Here we value real (investment) options in the presence of costly controls (acts of intervention) with random outcome (learning), in contrast to the practically standard *wait-and-see* approach. We extend the method proposed in Martzoukos (1998) by allowing the presence of multiple state-variables (instead of a single), which are assumed to follow not only GBM but also jump-diffusion processes. The assumption of multiple state-variables or asset prices has long been considered important as we know from the economics literature (i.e., Roberts and Weitzman, 1981, Gallini and Kotowitz, 1988) and as we can also see in the most recent real options literature (Childs et. al., 1998, Childs and Triantis, 1999). Jump-diffusion assumptions are important when there exist multiple classes of rare events, and each class can affect one or several state-variables, with these effects described by the joint probability distribution for each class of events. The importance of multiple classes (sources) of rare events has been identified in Martzoukos and Trigeorgis (1999) who value real options with a single state-variable. Such events are due for example to technological innovations (see Greenwood et. al., 1997), competitive entry (see Dunne et. al., 1988, and Ghemawat and Kennedy, 1999), regulatory uncertainty (see Brennan and Shwartz, 1982, and Teisberg, 1993), political uncertainty (Wagner, 1997), etc., events that can affect several of the state-variables. In addition to the randomly arriving events, there exist controls that if and when activated they can affect one or several (or all) state-variables, with the effect of each control on the state-variables described by a joint probability distribution. Thus, the joint impact of management´s actions (of value enhancement or information acquisition) on several variables or real assets is captured. We assume independence between controls, and also between controls and the state-variables. We also assume that the rare events and the controls carry non-systematic (non-priced) risk only. We wish to price such real options (with optimal activation of controls) for the entity that has monopoly power over these options.

We proceed as follows. First we develop the methodological framework for random controls and provide solutions when the state-variables follow GBM. Then we extend the framework to the case of jump-diffusion processes. To facilitate the exposition of the latter case, we first demonstrate how to price multivariate contingent claims with jump-diffusion processes in the absence of controls, and then real options in the presence of both optimally activated controls, and randomly arriving rare events.

II. The Optimization Problem

We will price claims contingent on several stochastic state-variables S_i, $i = (1,\ldots, N)$, which follow a process of the form

$$dS_i = (\mu_i - \delta_i)S_i dt + \sigma_i S_i dZ_i + S_i \sum_c k_{ic} dq_c \qquad (1a)$$

under the real probability measure, and

$$dS_i = (r - \delta_i)S_i dt + \sigma_i S_i dZ_i + S_i \sum_c k_{ic} dq_c \qquad (1b)$$

under the risk-neutral (see Constantinides, 1978, Harrison and Pliska, 1981, Cox, Ingersoll and Ross, 1985) probability measure. For real options, the dividend yield in the drift component is interpreted as in McDonald and Siegel (1984), or alternatively as in Brennan (1991). The *dZ* term denotes the increment to the standard Wiener process with $\sigma_i S_i$ the instantaneous standard deviation of the i^{th} diffusion process. The counter increment dq_c takes the value of zero before and one after a control is activated, and is a *control* variable (unlike the case of jump-diffusion where *dq* is a random variable). There exist $c = (1,\ldots, K)$ controls available, and when the control is activated, it affects each state-variable S_i proportionately (S_i is multiplied by $1 + k_{ic}$). The joint distribution of the $i = (1,\ldots, N)$ element vector $1 + k_{ic}$ can take many plausible forms (for each control). Here we assume that it is (multivariate) log-normal:

$$\ln(1 + k_{ic}) \sim \boldsymbol{\Phi_c}(\gamma_{ic} - .5\eta_{ic}^2, \eta_{ic}^2, \underline{\mathrm{P}}_c)$$

and

$$E[k_{ic}] = \exp(\gamma_{ic}) - 1$$

with ln denoting the natural logarithm, E[.] the expectations operator, and $\boldsymbol{\Phi_c}(\gamma_{ic} - .5\eta_{ic}^2, \eta_{ic}^2, \underline{\mathrm{P}}_c)$ the multivariate normal density function with vector of means $\gamma_{ic} - .5\eta_{ic}^2$, vector of variances η_{ic}^2, and correlation matrix $\underline{\mathrm{P}}_c$. Note the critical assumption that controls carry non-systematic risk, and are independent of each other and of the diffusion processes. The real claim *F* is contingent on multiple variables and controls and the cost $cst(c)$ of activating each control. Activation is optimal as the solution to the optimization problem defined by equations (2a) - (2d)

$$\underset{c,\ t(c)}{Maximize}[F(t, S_1,\ldots, S_N, cst(c))] \qquad (2a)$$

subject to:

$$dS_i = (r - \delta_i) S_i dt + \sigma_i S_i dZ_i + S_i \sum_c k_{ic} dq_c \qquad (2b)$$

with $i = (1,\ldots, N)$, and $\underline{P}$ the $[N\text{x}N]$ correlation matrix of the dZ_i terms;

the joint distributions of controls: $\ln(1 + k_{ic}) \sim \boldsymbol{\Phi}_c(\gamma_{ic} - .5\eta_{ic}^2, \eta_{ic}^2, \underline{P}_c)$ (2c)

with $\underline{P}_c$ the $[N\text{x}N]$ correlation matrices of the impact of the controls,

$c = (1,\ldots, K)$, and where $E[k_{ic}] = \exp(\gamma_{ic}) - 1$;

and the cost $cst(c)$ for each activated control. (2d)

In addition to the above, suitable boundary conditions (terminal, etc.) define properly the real option, which can be of the European or the American type, call or put. Other boundary conditions can define a complex multi-stage (i.e., compound, see Geske, 1979, and Kemna, 1993) investment opportunity.

A special case with analytic solution is the European option on the maximum or minimum of two assets (see Stulz, 1982) given activation of control(s) at time zero. The solution is isomorphic to Stulz (1982). The main differences are that the standard deviations of the two assets are replaced by the *conditional* standard deviations, the dividend yields are replaced by the *conditional* dividend yields, and their correlation is replaced by the *conditional* correlation (all *conditional* on control activation). For example, Stulz gives the European call on the maximum of two assets (which is directly extended to the case of assets paying dividend yields δ_1, and δ_2) and we denote as

$$\mathrm{E}_{\mathrm{CX}}(S_1, S_2, T, \delta_1, \delta_2, \sigma_1, \sigma_2, \rho_{1,2})$$

with $\rho_{1,2}$ the instantaneous correlation between dZ_1 and dZ_2, and T the time to option maturity. The same solution algorithm applies conditional on activation of i controls, with the following modifications of the input parameters (volatilities and correlations), and making use of the critical assumption that the controls are independent of each other and of the diffusion processes:

$$\mathrm{E}_{\mathrm{CX}}^{\mathrm{C}} = \tag{3}$$

$$\mathrm{E}_{\mathrm{CX}}\left(S_1, S_2, T, \delta_1(c) = \delta_1 - \sum_c \frac{\gamma_{1c}}{T}, \delta_2(c) = \delta_2 - \sum_c \frac{\gamma_{2c}}{T}, \sigma_1(c) = \sqrt{\sigma_1{}^2 + \sum_c \frac{\eta_{1c}{}^2}{T}}, \right.$$

$$\left. \sigma_2(c) = \sqrt{\sigma_2{}^2 + \sum_c \frac{\eta_{2c}{}^2}{T}}, \rho_{1,2}(c) = \frac{\sigma_{1,2} + \sum_c \frac{\eta_{1c}\eta_{2c}\rho_{1,2,c}}{T}}{\sigma_1(c)\ \sigma_2(c)} \right)$$

where $\sigma_{1,2}$ is the instantaneous covariance of the logarithms of the two assets due to the diffusion processes, $\delta_1(c)$, $\delta_2(c)$, $\sigma_1(c)$, $\sigma_2(c)$ and $\rho_{1,2}(c)$ are the conditional dividend yields, the conditional standard deviations, and the conditional correlation given activation of controls. In the same way we can extend the other models (call on the minimum, and put on the maximum or the minimum) in the presence of controls activated at time zero. For the case of N state-variables, the Johnson (1987) option model on several assets would be similarly extended conditional on activation of controls at time zero, only all pair-wise correlations would require similar adjustments.

For the most general solution to this control problem we will develop a multi-dimensional Markov-chain numerical solution method that is an extension of the random control method, proposed in Martzoukos (1998) in the presence of a single state-variable following GBM. The Markov-chain will augment the Boyle, Evnine and Gibbs (1989) (or alternatively any other suitable method for multi-dimensional contingent claims, like Kamrad and Ritchken, 1991, and Ekvall, 1996) to value the real option in the case of activated controls.

One step later and due to the control alone, the joint probability $\text{Prob}_C\{\underline{n}\}$ that the state-variables $\underline{S}(t)$ move to $\underline{S}(t + \Delta t)$ with a vertical transition of $\underline{n} = [n_1,\ldots, n_N]$ steps, is multivariate-normal and is defined as

$$\text{Prob}_C\{\underline{n}\} = \tag{4}$$

$$\text{Prob}_C\left\{\ln\left[\frac{S_1(t+\Delta t)}{S_1(t)}\right] = n_1\sigma_1\sqrt{\Delta t}, \right.$$
$$\ln\left[\frac{S_2(t+\Delta t)}{S_2(t)}\right] = n_2\sigma_2\sqrt{\Delta t},$$
$$\left. \ldots,\quad \ln\left[\frac{S_N(t+\Delta t)}{S_N(t)}\right] = n_N\sigma_N\sqrt{\Delta t}\right\}$$

since in the Boyle, Evnine, and Gibbs (1989) method that we implement, each movement of the logarithm of the state-variable S is either up by $\sigma \sqrt{(\Delta t)}$, or down by $-\sigma \sqrt{(\Delta t)}$. But due to the geometric Brownian motion alone, the vector $\underline{S}(t)$ can move on the solution grid to $\underline{S}(t + \Delta t)$ by either $n_i = m_i = 1$ or $n_i = m_i = -1$ for each variable i, with probabilities

$$P_D\{m_1,m_2,\ldots,m_N\}$$

that exhaust all pairwise combinations of one up ($m = 1$) or one down ($m = -1$) move:

$$P_D\{1,1,1,\ldots,1\},\ P_D\{-1,1,1,\ldots,1\},\ P_D\{1,-1,1,\ldots,1\},\ P_D\{-1,-1,1,\ldots,1\},\ \text{etc.}$$

and are given in Boyle, Evnine and Gibbs (1989). Subsequently, the probabilities due to both the control and the diffusion processes equal

$$\text{Prob}\{\underline{n}\} = \sum_{\underline{m}}\left\{P_D\{\underline{m}\}\text{Prob}_C\{\underline{n}-\underline{m}\}\right\} \tag{5}$$

with summation over all possible realizations of vector $\underline{m}$.

Let us see the implementation for the more tractable case of $N = 2$. Due to the control alone, the joint probability $\mathrm{Prob}_C\{\underline{n}\}$ that the state-variables $\underline{S}(t)$ move on the solution grid to $\underline{S}(t + \Delta t)$ by $\underline{n} = [n_1, n_2]$ steps, is given by

$$\mathrm{Prob}_C\{n_1, n_2\} = \tag{6}$$

$$\mathrm{Prob}_C\left\{\ln\left[\frac{S_1(t+\Delta t)}{S_1(t)}\right] = n_1\sigma_1\sqrt{\Delta t},\ \ \ln\left[\frac{S_2(t+\Delta t)}{S_2(t)}\right] = n_2\sigma_2\sqrt{\Delta t}\right\}$$

and due to the geometric Brownian motion alone, the vector $\underline{S}(t)$ can move on the solution grid to $\underline{S}(t + \Delta t)$ by $m = [1,1]$, $m = [1,-1]$, $m = [-1,-1]$, or $m = [-1,1]$, with all probabilities

$$\mathrm{P_D}\{m_1,m_2\} = \mathrm{P_D}\{\underline{m}\}$$

again given from the results in Boyle, Evnine and Gibbs (1989).

The probabilities due to both the diffusion processes and the control are given by

$$\mathrm{Prob}\{\underline{n}\} = \sum_{\underline{m}}\left\{\mathrm{P_D}\{\underline{m}\}\mathrm{Prob}_C\{\underline{n}-\underline{m}\}\right\} = \tag{7}$$

$$\mathrm{P_D}\{1{,}1\}\mathrm{Prob}_C\{\underline{n}-1\} + \mathrm{P_D}\{-1{,}-1\}\mathrm{Prob}_C\{\underline{n}+1\}$$

$$+\,\mathrm{P_D}\{1{,}-1\}\mathrm{Prob}_C\{n_1-1, n_2+1\} + \mathrm{P_D}\{1{,}-1\}\mathrm{Prob}_C\{n_1-1, n_2+1\}$$

For implementation purposes, each probability $\mathrm{Prob}_C\{n_1, n_2\}$ is approximated by

$$\mathrm{Prob}_{\mathrm{C}}\{n_1, n_2\} \cong \tag{8}$$

$$\mathrm{BIV}_{\mathrm{C}}\{n_1 + .5, n_2 + .5, \rho_{1,2,c}\} + \mathrm{BIV}_{\mathrm{C}}\{n_1 - .5, n_2 - .5, \rho_{1,2,c}\}$$

$$- \mathrm{BIV}_{\mathrm{C}}\{n_1 - .5, n_2 + .5, \rho_{1,2,c}\} - \mathrm{BIV}_{\mathrm{C}}\{n_1 + .5, n_2 - .5, \rho_{1,2,c}\}$$

where $\mathrm{BIV}_{\mathrm{C}}$ is the cumulative bivariate normal with subscript C denoting movement due to the control alone. Any term of the form $\mathrm{BIV}_{\mathrm{C}}\{n_1, n_2, \rho_{1,2,c}\}$ is evaluated (after standardization) from the cumulative bivariate standard normal

$$\mathrm{BIV}_{\mathrm{C}}\{n_1, n_2, \rho_{1,2,c}\} = \tag{9}$$

$$\mathrm{BIVS}_{\mathrm{C}}\left\{\frac{n_1\sigma_1\sqrt{\Delta t} - (\gamma_{1c} - .5\eta_{1c}{}^2)}{\eta_{1c}},\ \frac{n_2\sigma_2\sqrt{\Delta t} - (\gamma_{2c} - .5\eta_{2c}{}^2)}{\eta_{2c}},\ \rho_{1,2,c}\right\}$$

where the term $\rho_{1,2,c}$ is the correlation of the logarithms of S_1 and S_2 due to activation of control c.

Discussion and Numerical Results

Through the analytic solution in eq. (3) for the value of a call option on the maximum of two assets, we provide the results shown in Tables 1 and 2. In the first Table we see clearly the striking difference in real option values between the case of absence of control (figures in bold) and when a control is present. Note that the controls used in these examples have an expected impact on the underlying assets $\gamma_c = 0$ (pure learning actions).

Insert Tables 1 and 2 about here

The simple implementation of real option models can severely underestimate option values and thus lead to erroneous decision making (in exercise or purchase/sale of investment rights). In the first table we also see the comparison when the control either affects only one of the underlying assets, or it affects both. Actions affecting simultaneously more than one variable (or asset) clearly are much more valuable. In the second Table we see that in relative terms, more value is added when the options are out-of-the-money than when they are in-the-money. Control/learning is much more important for investment opportunities that are not profitable yet. In the same Table we also compare the impact of a single or of two controls. By neglecting the availability of such actions, we can easily underprice the (out-of-the-money) real option by a factor of 1/510 (single control) and 1/1073 (two controls) for the parameter values chosen.

In Tables 3 and 4 we see the results from the implementation of the numerical scheme described by eq. (6) – (9) and defined in eq. (5).

Insert Tables 3 and 4 about here

The accuracy of the numerical scheme is investigated in Table 3 by comparing numerical results with those provided by implementation of the analytic solution in eq. (3). The numerical error most often does not exceed the error caused by the lattice of our choice (the Boyle, Evnine, and Gibbs, 1989) as shown by the figures in bold. In general the accuracy provided is considered very satisfactory for both shorter (T = 1 year) and longer (T = 5 years) maturities. Table 4 provides a comparison of the option values with a single and with two *costly* controls, where in the second case the second control is timed beyond t = 0 (at t = T/2). The values at the top provide the upper bound (when controls are costless) and the values in bold at the bottom the lower bound (when controls are nonexistent or extremely costly). The values in the left column could also be calculated with the analytic model, since the single control is exercised at time $t = 0$. In that case, the optimal option value is the maximum of the value in the absence of control, or the value *conditional* on control activation minus the cost of the control. For cases like in the second column where several optional actions of value enhancement or information acquisition are *embedded* at time $t \neq 0$ there exists no analytic solution even for the simple call option on the maximum of two assets. Actions of intervention sequentially spaced in time (and most often costly) are the most generally appearing and the numerical solution method is necessary in order to solve for the option value under optimal control activation. Treatment of realistic problems require the valuation of complex real options (i.e., Kemna, 1993, and Trigeorgis, 1993), for which, even in the absence of controls, no analytic solution would exist anyway. In conclusion, optimal valuation and decision making requires (often numerical) valuation of real options in the presence of control/learning actions, where optimal control activation depends on the trade-off between value added due to control activation, and the control´s cost.

III. Extension: Random Controls for Multivariate Jump-Diffusion Processes

We will extend the method of the previous section when the state-variables follow discontinuous processes (jump-diffusion). The random jump arrivals follow Poisson distributions with given frequencies, and in the most general case have non-constant jump size. We will retain the assumption that the distribution of the jumps is log-normal. We will also work with the most general assumption that each jump affects all state-variables (whereas as a special case some jumps can affect only some of the state-variables). For expositional convenience (and because it has a sufficient interest by itself), we will first treat the valuation problem in the absence of controls. The method is close in spirit to the one developed in Amin (1993) (see also Bates 1991) for the case of one-dimensional jump-diffusion process, drawing on convergence properties studied in Kushner (1977), Kushner and DiMasi (1978), and Kushner (1990). The results are applicable to both financial and real option problems. Then we will value real investment options in the presence of both random controls and randomly arriving jumps (rare events).

Analytic Solutions and Numerical Approximations under the Multi-Dimensional Jump-Diffusion Process

We will demonstrate the Markov-chain numerical approximation method in the absence of control as an extension of Amin (1993), and we will provide analytic solutions for options on several assets in the case of jump-diffusion processes. We assume the existence of several classes of jumps affecting each state-variable. Martzoukos and Trigeorgis (1999) have demonstrated the importance of considering multiple classes (representing different sources) of randomly arriving rare events, and they have priced such claims for one state-variable. In our case and in the presence of several state-variables (this section draws on Martzoukos, 2000), we have under the risk-neutral probability measure

$$dS_i = (r - \delta_i - \sum_j \lambda_j \bar{k}_{ij})S_i dt + \sigma_i S_i dZ_i + S_i \sum_j k_{ij} dq_j \tag{10}$$

with $i = (1,\ldots, N)$ state-variables, and $j = (1,\ldots, L)$ classes of randomly arriving jumps that are Poisson distributed with yearly frequency λ_j. Jumps of different classes are independent of each other and of the GBM. We retain the assumption that jumps carry non-systematic risk only. We allow jumps to affect in the most general case all state-variables (not necessarily to the same extent), having a multivariate log-normal distribution across state-variables:

$$\ln(1 + k_{ij}) \sim \boldsymbol{\Phi}_j(\gamma_{ij} - .5\eta_{ij}^2, \eta_{ij}^2, \underline{\mathrm{P}}_j) \tag{11}$$

$$\text{with } E[k_{ij}] = \exp(\gamma_{ij}) - 1$$

$$\text{and the } [N\text{x}N] \text{ correlation matrices } \underline{\mathrm{P}}_j,\ j = (1,\ldots, L)$$

Analytic solutions are rare and can be useful to solve special cases of financial option models, but most real option situations would require more complex analysis without analytic solutions. We demonstrate the analytic solution for the case of a European option on several assets. In the absence of jumps, Johnson (1987) gives the European option on several assets. Here we will demonstrate for simplicity how to extend the Stulz (1982) option model on two assets to the case of jump-diffusion, and the Johnson model can be extended similarly. The European call on the maximum in the absence of jumps is denoted

$$\mathrm{E}_{\mathrm{CX}}(S_1, S_2, T, \delta_1, \delta_2, \sigma_1, \sigma_2, \rho_{1,2})$$

In the presence of jumps, the solution is given by an iterated integral. Iterations are conditional on jump realizations where the vector $\underline{l} = (l_1,\ldots,l_L)$ presenting the number of realized jumps in each of the L jump classes. The solution is

$$\mathrm{E}^{\mathrm{J}}_{\mathrm{CX}} = \sum_{l_1=0}^{\infty}\cdots\sum_{l_L=0}^{\infty}\mathrm{Prob}\{l_1,\ldots,l_L\}\mathrm{E}_{\mathrm{CX}}[S_1, S_2, T, \delta_1(\underline{l}), \delta_2(\underline{l}), \sigma_1(\underline{l}), \sigma_2(\underline{l}), \rho_{1,2}(\underline{l})] \quad (12)$$

with

$$\delta_1(\underline{l}) = \delta_1 + \sum_j \lambda_j \bar{k}_{1j} - \sum_j \frac{l_j \gamma_{1j}}{T}$$

$$\delta_2(\underline{l}) = \delta_2 + \sum_j \lambda_j \bar{k}_{2j} - \sum_j \frac{l_j \gamma_{2j}}{T}$$

$$\sigma_1(\underline{l}) = \sqrt{\sigma_1^{\ 2} + \sum_j \frac{l_j \eta_{1j}^{\ 2}}{T}}$$

$$\sigma_2(\underline{l}) = \sqrt{\sigma_2^{\ 2} + \sum_j \frac{l_j \eta_{2j}^{\ 2}}{T}}$$

and

$$\rho_{1,2}(\underline{l}) = \frac{\sigma_{1,2} + \sum_j \frac{l_j \eta_{1j} \eta_{2j} \rho_{1,2,j}}{T}}{\sigma_1(\underline{l})\ \sigma_2(\underline{l})}$$

Given that

$$\text{Prob}\{l_j\} = e^{-\lambda_j T}\ \frac{(\lambda_j T)^{l_j}}{l_j!} \tag{13a}$$

and due to the independence of the *j* rare events,

$$\text{Prob}\{l_1, ..., l_L\} = \text{Prob}\{\underline{l}\} = \prod_j \left[e^{-\lambda_j T}\ \frac{(\lambda_j T)^{l_j}}{l_j!} \right]. \tag{13b}$$

The provision of the following Markov-chain probabilities (like in the previous section) can accommodate the treatment of the most general problems by augmenting the Boyle, Evnine, and Gibbs method. We use the same definitions with section II, only Prob_C is replaced with the similarly defined Prob_J but with random (resulting from the jump-diffusion assumption) and not with controlled arrival. The transition probabilities are now given by

$$\text{Prob}\{\underline{n}\} =$$

$$\text{Prob}\{\underline{l} = 0\}\text{P}_\text{D}\{\underline{n}\} + \sum_j \left\{ \text{Prob}\{l_{k=j} = 1, l_{k \neq j} = 0\} \sum_{\underline{m}} \left\{ \text{P}_\text{D}\{\underline{m}\} \text{Prob}_{\text{J}(j)}\{\underline{n} - \underline{m}\} \right\} \right\} \tag{14a}$$

with the first term of the RHS implying that the position is attainable through the GBM alone (so $\underline{n}$ is a subset of the possible realizations of $\underline{m}$), and when this is not feasible,

$$\text{Prob}\{\underline{n}\} = \sum_j \left\{ \text{Prob}\{l_{k=j} = 1, l_{k \neq j} = 0\} \sum_{\underline{m}} \left\{ \text{P}_\text{D}\{\underline{m}\} \text{Prob}_{\text{J}(j)}\{\underline{n} - \underline{m}\} \right\} \right\} \tag{14b}$$

with the parenthesis in $\text{Prob}_{\text{J}(j)}$ signifying that calculations are dependent on the distributional characteristics of rare event *j*.

Like in the previous section for Prob_D, in the two-dimensional case we calculate

$$\sum_{\underline{m}} \left\{ \text{P}_\text{D}\{\underline{m}\} \text{Prob}_{\text{J}(j)}\{\underline{n} - \underline{m}\} \right\} = \tag{15}$$

$$\text{P}_\text{D}\{1,1\}\text{Prob}_{\text{J}(j)}\{\underline{n} - 1\} + \text{P}_\text{D}\{-1,-1\}\text{Prob}_{\text{J}(j)}\{\underline{n} + 1\}$$

$$+ \text{P}_\text{D}\{1,-1\}\text{Prob}_{\text{J}(j)}\{n_1 - 1, n_2 + 1\} + \text{P}_\text{D}\{1,-1\}\text{Prob}_{\text{J}(j)}\{n_1 - 1, n_2 + 1\}$$

and for implementation purposes each probability $\text{Prob}_{\text{J}(j)}\{n_1, n_2\}$ is approximated by

$$\text{Prob}_{\text{J}(j)}\{n_1, n_2\} \cong \tag{16}$$

$$\text{BIV}_{\text{J}(j)}\{n_1 + .5, n_2 + .5, \rho_{1,2,j}\} + \text{BIV}_{\text{J}(j)}\{n_1 - .5, n_2 - .5, \rho_{1,2,j}\}$$

$$- \text{BIV}_{\text{J}(j)}\{n_1 - .5, n_2 + .5, \rho_{1,2,j}\} - \text{BIV}_{\text{J}(j)}\{n_1 + .5, n_2 - .5, \rho_{1,2,j}\}$$

with the cumulative bivariate normal calculated after standardization like in section II (making use of the distributional characteristics of the random events instead of controls).

Random Controls under Jump-Diffusion Assumptions

Now we are ready to combine the control method of section II with our results on multi-dimensional jump-diffusion. The state-variables follow under the risk-neutral measure the process

$$dS_i = (r - \delta_i - \sum_j \lambda_j \bar{k}_{ij})S_i dt + \sigma_i S_i dZ_i + S_i \sum_j k_{ij} dq_j + S_i \sum_c k_{ic} dq_c \qquad (17)$$

where dq_j is a random (Poisson distributed) variable, and dq_c is a control variable. Optimal activation of controls is a result to the optimization problem defined by equations (18a) – (18e)

$$\underset{c,\ t(c)}{Maximize}[F(t, S_1, ..., S_N, cst(c))] \qquad (18a)$$

subject to:

$$dS_i = (r - \delta_i - \sum_j \lambda_j \bar{k}_{ij})S_i dt + \sigma_i S_i dZ_i + S_i \sum_j k_{ij} dq_j + S_i \sum_c k_{ic} dq_c \qquad (18b)$$

with $i = (1,\dots, N)$, and $\underline{P}$ the $[N\text{x}N]$ correlation matrix of the dZ_i terms;

the joint distributions of rare events: $\ln(1 + k_{ij}) \sim \boldsymbol{\Phi_j}(\gamma_{ij} - .5\eta_{ij}^2, \eta_{ij}^2, \underline{P}_j)$ (18c)

with $j = (1,\dots, L)$, $\underline{P}_j$ the $[N\text{x}N]$ correlation matrices of the impact of jumps,

and where $E[k_{ij}] = \exp(\gamma_{ij}) - 1$;

the joint distributions of controls: $\ln(1 + k_{ic}) \sim \boldsymbol{\Phi_c}(\gamma_{ic} - .5\eta_{ic}^2, \eta_{ic}^2, \underline{P}_c)$ (18d)

with $c = (1,\dots, K)$, $\underline{P}_c$ the $[N\text{x}N]$ correlation matrices of the impact of controls,

and where $E[k_{ic}] = \exp(\gamma_{ic}) - 1$;

and the cost $cst(c)$ for each activated control. (18e)

An analytic solution exists again for the European call option on the maximum of two assets in the presence of jump-diffusion and *conditional* on activated controls. The solution is the iterated integral

$$\mathrm{E}_{\mathrm{CX}}^{\mathrm{J,C}} = \tag{19}$$

$$\sum_{l_1=0}^{\infty}\ldots\sum_{l_L=0}^{\infty}\left\{\mathrm{Prob}\{l_1,\ldots,l_L\}\mathrm{E}_{\mathrm{CX}}[S_1,S_2,T,\delta_1(\underline{l},c),\delta_2(\underline{l},c),\sigma_1(\underline{l},c),\sigma_2(\underline{l},c),\rho_{1,2}(\underline{l},c)\right\}$$

with

$$\delta_1(\underline{l},c)=\delta_1+\sum_j\lambda_j\bar{k}_{1j}-\sum_j\frac{l_j\gamma_{1j}}{T}-\sum_c\frac{\gamma_{1c}}{T}$$

$$\delta_2(\underline{l},c)=\delta_2+\sum_j\lambda_j\bar{k}_{2j}-\sum_j\frac{l_j\gamma_{2j}}{T}-\sum_c\frac{\gamma_{2c}}{T}$$

$$\sigma_1(\underline{l},c)=\sqrt{{\sigma_1}^2+\sum_j\frac{l_j{\eta_{1j}}^2}{T}+\sum_c\frac{{\eta_{1c}}^2}{T}}$$

$$\sigma_2(\underline{l},c)=\sqrt{{\sigma_2}^2+\sum_j\frac{l_j{\eta_{2j}}^2}{T}+\sum_c\frac{{\eta_{2c}}^2}{T}}$$

and

$$\rho_{1,2}(\underline{l},c)=\frac{\sigma_{1,2}+\sum_j\dfrac{l_j\eta_{1j}\eta_{2j}\rho_{1,2,j}}{T}+\sum_c\dfrac{\eta_{1c}\eta_{2c}\rho_{1,2,c}}{T}}{\sigma_1(\underline{l},c)\ \sigma_2(\underline{l},c)}.$$

Similarly we can derive the option on several assets.

In order to solve numerically the most general problem, we now provide the transition probabilities for a multivariate jump-diffusion process under activation of a random control

$$\mathrm{Prob}\{\underline{n}\} = \tag{20}$$

$$\text{Prob}\{\underline{l}=0\}\sum_{\underline{m}}\left\{\text{P}_{\text{D}}\{\underline{m}\}\text{Prob}_{\text{C}}\{\underline{n}-\underline{m}\}\right\}+$$

$$\sum_{j}\left\{\text{Prob}\{l_{k=j}=1,l_{k\neq j}=0\}\sum_{\underline{m}}\left\{\text{P}_{\text{D}}\{\underline{m}\}\text{Prob}_{\text{J}(j),\text{C}}\{\underline{n}-\underline{m}\}\right\}\right\}$$

with $\text{Prob}_{\text{J}(j),\text{C}}$ denoting the joint impact of a random event of class j and the activated control. The special case with two state-variables is given and approximated again similarly with section II. The definition of

$$\sum_{\underline{m}}\left\{\text{P}_{\text{D}}\{\underline{m}\}\text{Prob}_{\text{C}}\{\underline{n}-\underline{m}\}\right\}$$

is exactly like in section II, and similarly we define

$$\sum_{\underline{m}}\left\{\text{P}_{\text{D}}\{\underline{m}\}\text{Prob}_{\text{J}(j),\text{C}}\{\underline{n}-\underline{m}\}\right\}.$$

The joint probabilities resulting from activation of the control in the absence of jumps are approximated by

$$\text{Prob}_{\text{C}}\{n_1,n_2\}\cong \tag{21}$$

$$\text{BIV}_{\text{C}}\{n_1+.5,n_2+.5,\rho_{1,2,c}\}+\text{BIV}_{\text{C}}\{n_1-.5,n_2-.5,\rho_{1,2,c}\}$$

$$-\text{BIV}_{\text{C}}\{n_1-.5,n_2+.5,\rho_{1,2,c}\}-\text{BIV}_{\text{C}}\{n_1+.5,n_2-.5,\rho_{1,2,c}\}$$

with

$$\text{BIV}_{\text{C}}\{n_1,n_2,\rho_{1,2,c}\}= \tag{22}$$

$$\text{BIVS}_{\text{C}}\left\{\frac{n_1\sigma_1\sqrt{\Delta t}-(\gamma_{1c}-.5\eta_{1c}{}^2)}{\eta_{1c}},\ \frac{n_2\sigma_2\sqrt{\Delta t}-(\gamma_{2c}-.5\eta_{2c}{}^2)}{\eta_{2c}},\ \rho_{1,2,c}\right\}.$$

The joint probabilities resulting from the simultaneous impact of control activation and a jump are approximated by

$$\mathrm{Prob}_{\mathrm{J}(j),\mathrm{C}}\{n_1, n_2\} \cong \tag{23}$$

$$\mathrm{BIV}_{\mathrm{J}(j),\mathrm{C}}\{n_1+.5, n_2+.5, \rho_{1,2}(\mathrm{J}(j),c)\} + \mathrm{BIV}_{\mathrm{J}(j),\mathrm{C}}\{n_1-.5, n_2-.5, \rho_{1,2}(\mathrm{J}(j),c)\}$$

$$-\,\mathrm{BIV}_{\mathrm{J}(j),\mathrm{C}}\{n_1-.5, n_2+.5, \rho_{1,2}(\mathrm{J}(j),c)\} - \mathrm{BIV}_{\mathrm{J}(j),\mathrm{C}}\{n_1+.5, n_2-.5, \rho_{1,2}(\mathrm{J}(j),c)\}$$

with

$$\mathrm{BIV}_{\mathrm{J}(j),\mathrm{C}}\{n_1, n_2, \rho_{1,2}(\mathrm{J}(j),c)\} = \tag{24}$$

$$\mathrm{BIVS}_{\mathrm{C}}\left\{\frac{n_1\sigma_1\sqrt{\Delta t}-(\gamma_{1c}-.5\eta_{1c}{}^2)-(\gamma_{1j}-.5\eta_{1j}{}^2)}{\sqrt{\eta_{1c}{}^2+\eta_{1j}{}^2}},\right.$$

$$\left.\frac{n_2\sigma_2\sqrt{\Delta t}-(\gamma_{2c}-.5\eta_{2c}{}^2)-(\gamma_{2j}-.5\eta_{2j}{}^2)}{\sqrt{\eta_{2c}{}^2+\eta_{2j}{}^2}},\ \rho_{1,2}(\mathrm{J}(j),c)\right\}$$

and

$$\rho_{1,2}(\mathrm{J}(j),c) = \frac{\eta_{1c}\eta_{2c}\rho_{1,2,c}+\eta_{1j}\eta_{2j}\rho_{1,2,j}}{\sqrt{\eta_{1c}{}^2+\eta_{1j}{}^2}\sqrt{\eta_{2c}{}^2+\eta_{2j}{}^2}}\ .$$

IV. Conclusions

Martzoukos (1998) proposed the *random controls* methodology where controlled jumps of random size represent acts of management intervention. Such acts have uncertain outcome, the realization of which represents learning. The method deviates from the practically standard approach in the real options literature of *waiting-to-see* and recognizes that a large component of learning comes from deliberate actions, often at a considerable cost.

Optimal decision making is the result of a maximization problem where the trade-off between the cost of control/learning and the value added by such actions is explicitly taken into consideration.

In this paper the methodology is generalized from one dealing with the special case of actions affecting only one state-variable, to actions that affect simultaneously several. Thus we explicitly recognize that more than one uncertainty is often present. Furthermore, development of management plans about future investments should recognize before investment decision is made the flexibility options of retaining alternative courses of action (i.e., mutually exclusive investment alternatives). And after investment is made, such plans should recognize the flexibility options of switching modes of operation (i.e., to alternative output or input sources). Overall uncertainty is much more complex than what one state-variable can capture. Acts of management intervention (control/learning) is thus directed towards more than one state-variable. Managerial intervention in the presence of several flexibility options is in reality very important (a result consistent with the insights in Roberts and Weitzman, 1981). The value of keeping several (real) options alive is also clearly recognized by industry, as for example in *Exploiting Uncertainty* (a recent Business Week, 1999, article). Thus, control/learning is more fruitful if directed to more than just one of them.

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TABLE 1

THE VALUE OF AN OPTION ON 2 ASSETS IN THE PRESENCE OF A CONTROL

		Impact of a control affecting *one* or *both* variables			
		One	*both*		
	η_c =	ρ_c = ------	ρ_c = -0.50	ρ_c = 0.00	ρ_c = 0.50
T = 1	0.30	14.2306	22.6419	21.0118	18.7334
	0.20	10.7376	15.7534	14.6691	13.2443
	0.10	7.6796	9.5295	9.0565	8.4993
	------	**6.1887**	**6.1887**	**6.1887**	**6.1887**
T = 5	0.30	16.0208	21.3874	20.1738	18.6078
	0.20	13.7583	16.6422	15.9109	15.0455
	0.10	12.1042	12.9702	12.7225	12.4585
	------	**11.4727**	**11.4727**	**11.4727**	**11.4727**

Notes: Underlying assets $S_1 = S_2 = 100$, exercise price X = 100, option maturity T = 1 or T = 5 years, with standard deviations $\sigma_1 = \sigma_2 = 0.10$ and $\rho_{1,2} = 0.25$, dividend yields $\delta_1 = \delta_2 = 0.05$, and riskless rate r = 0.05.

Both control volatilities equal η_c, with correlation ρ_c, both expected changes $\gamma_c = 0.0$, exercise time t = 0, and control cost *cst* = 0.0.

TABLE 2

THE VALUE OF AN OPTION ON 2 ASSETS IN THE PRESENCE OF CONTROLS

		Impact on *in-* and *out-of-the-money* options		
	η_c =	$S_1 = S_2 = 75$	$S_1 = S_2 = 100$	$S_1 = S_2 = 125$
One control	0.30	4.8984	21.0118	45.4242
	0.20	1.6741	14.6691	38.5876
	0.10	0.1782	9.0565	32.6647
	------	**0.0096**	**6.1887**	**29.5885**
Two controls	0.30	10.2973	29.2667	54.5528
	0.20	4.2527	19.8975	44.2100
	0.10	0.5628	11.2297	34.9530
	------	**0.0096**	**6.1887**	**29.5885**

Notes: Underlying assets $S_1 = S_2$, exercise price X = 100, option maturity T = 1 years, with standard deviations $\sigma_1 = \sigma_2 = 0.10$ and $\rho_{1,2} = 0.25$, dividend yields $\delta_1 = \delta_2 = 0.05$, and riskless rate r = 0.05.

The volatilities of both controls equal η_c, with correlation $\rho_c = 0.0$, both expected changes $\gamma_c = 0.0$, exercise times t = 0, and control costs *cst* = 0.0.

TABLE 3

ACCURACY OF THE 2-D MARKOV-CHAIN NUMERICAL IMPLEMENTATION

	ANALYTIC	NUMERICAL
T = 1 years		
η_c = 0.30, ρ_c = -0.50	22.6419	22.6380
η_c = 0.30, ρ_c = 0.00	21.0118	21.0098
η_c = 0.30, ρ_c = 0.50	18.7334	18.7319
η_c = 0.20, ρ_c = -0.50	15.7534	15.7448
η_c = 0.20, ρ_c = 0.00	14.6691	14.6637
η_c = 0.20, ρ_c = 0.50	13.2443	13.2396
η_c = 0.10, ρ_c = -0.50	9.5295	9.5313
η_c = 0.10, ρ_c = 0.00	9.0565	9.0584
η_c = 0.10, ρ_c = 0.50	8.4993	8.5012
η_c = ----- (no control)	**6.1887**	**6.2084**
T = 5 years		
η_c = 0.30, ρ_c = -0.50	21.3874	21.3897
η_c = 0.30, ρ_c = 0.00	20.1738	20.1766
η_c = 0.30, ρ_c = 0.50	18.6078	18.6114
η_c = 0.20, ρ_c = -0.50	16.6422	16.6497
η_c = 0.20, ρ_c = 0.00	15.9109	15.9178
η_c = 0.20, ρ_c = 0.50	15.0455	15.0534
η_c = 0.10, ρ_c = -0.50	12.9702	12.9848
η_c = 0.10, ρ_c = 0.00	12.7225	12.7371
η_c = 0.10, ρ_c = 0.50	12.4585	12.4734
η_c = ----- (no control)	**11.4727**	**11.5103**

Notes: Underlying assets $S_1 = S_2 = 100$, exercise price X = 100, option maturity T = 1 or 5 years, with standard deviations $\sigma_1 = \sigma_2 = 0.10$ and $\rho_{1,2} = 0.25$, dividend yields $\delta_1 = \delta_2 = 0.05$, and riskless rate r = 0.05.

Both control volatilities equal η_c, with correlation ρ_c, and both expected changes $\gamma_c = 0.0$ and control cost *cst* = 0.0, and exercise time at t = 0.

The 2-D numerical method is an augmentation of Boyle, Evnine and Gibbs (1989) (n = 25 steps), making sure that the Markov-Chain probabilities sum (approximately) to unity.

TABLE 4

THE (NUMERICAL) 2-D MARKOV-CHAIN WITH *COSTLY* CONTROLS

	One control	Two controls
Control cost *cst* = 0.0		
ρ_c = -0.50	15.7448	21.2681
ρ_c = 0.00	14.6637	19.7628
ρ_c = 0.50	13.2396	17.6196
Control cost *cst* = 3.0		
ρ_c = -0.50	12.7448	15.7162
ρ_c = 0.00	11.6637	14.1864
ρ_c = 0.50	10.2396	12.0911
Control cost *cst* = 7.0		
ρ_c = -0.50	8.7448	9.4680
ρ_c = 0.00	7.6637	8.0810
ρ_c = 0.50	6.2396	6.6482
No control: (Control cost *cst* = ∞)	**6.2084**	**6.2084**

Notes: Underlying assets $S_1 = S_2$, X = 100, option maturity T = 1 years, with standard deviations $\sigma_1 = \sigma_2 = 0.10$ and $\rho_{1,2} = 0.25$, dividend yields $\delta_1 = \delta_2 = 0.05$, and riskless rate r = 0.05.

Controls have both volatilities $\eta_c = 0.20$, correlation ρ_c, control cost *cst*, and timing of controls is at t = 0 for a single control, and at t = 0 and t = T/2 for the case of two controls.

The 2-D method is an augmentation of Boyle, Evnine and Gibbs (1989) (n = 25 steps), making sure that the Markov-Chain probabilities sum (approximately) to unity.

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