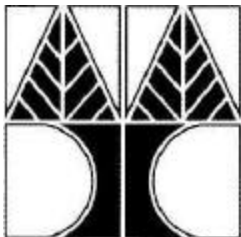




***RESOLVING A REAL OPTIONS PARADOX
WITH INCOMPLETE INFORMATION:
AFTER ALL, WHY LEARN?***

***by
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Resolving a Real Options Paradox with Incomplete Information: After All, *Why Learn?*

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Resolving a Real Options Paradox with Incomplete Information: After All, *Why Learn?*

Abstract

In this paper we discuss a real options paradox of managerial intervention directed towards learning and information acquisition: since options are in general increasing functions of volatility whereas learning reduces uncertainty, why would we want to learn? Examining real options with (costly) learning and path-dependency, we show that conditioning of information and optimal timing of learning leads to superior decision-making and enhances real option value.

Introduction

Most of the real options literature (see Dixit and Pindyck, 1994, and Trigeorgis, 1996) has examined the value of flexibility in investment and operating decisions, but little has been written about management's ability to intervene in order to change strategy or acquire information (learn). Majd and Pindyck (1989), and Pennings and Lint (1997) examine real options with passive learning, while Childs, et al. (1999), and Epstein, et al. (1999) use a filtering approach towards learning. The importance of learning actions like exploration, experimentation, and R&D was recognized early on in the economics literature (e.g., Roberts and Weitzman, 1981). Compound option models (Geske, 1977, Carr, 1988, and Paddock, Siegel, and Smith, 1988) capture some form of learning as the result of observing the evolution of a stochastic variable. Sundaresan (2000) recently emphasizes the need for adding an incomplete information framework to real options valuation problems.

Although many variables, like expected demand or price for a new product, are typically treated as observable (deterministic or stochastic), in many situations it is more realistic to assume that they are simply subjective estimates of quantities that will be actually observed or realized later. Our earlier estimates can thus change in unpredictable ways. Ex ante, their change is a random variable with (presumably) a known probability distribution. These are often price-related variables so in order to avoid negative values, it can be assumed that the relative change (one plus) has a (discrete or continuous) distribution that precludes negative values. Abraham and Taylor (1993) consider jumps at known times to capture additional uncertainty induced in option pricing due to foreseeable announcement events. Martzoukos (1998) examines real options with controlled jumps of random size (random controls) to model intervention of management as intentional actions with uncertain outcome. He assumes that such actions are independent of each other. Under incomplete information, costly control actions can improve estimates about important variables or parameters, either by eliminating or by reducing uncertainty.

This paper seeks to resolve an apparent paradox in real options valuation under incomplete information: Since (optional) learning actions intended to improve estimates actually reduce uncertainty, whereas option values are in general indecreasing functions of uncertainty, why would the decision-maker want to exercise the uncertainty-reducing learning options? By introducing a model of learning with path-dependency, we investigate the *optimal timing* of actions of information acquisition that result in reduction of uncertainty in order to enhance real option value.

If uncertainty is fully resolved, exercise of an investment option on stochastic asset S^* with exercise cost X yields $S^* - X$. If a learning action has not been taken before the investment decision is made, resolution of uncertainty (learning) would occur ex post. Ex ante, the investment decision must be made based solely on expected (instead of actual) outcomes, in which case exercise of the real option is expected to provide $E[S^*] - X$. For tractability, we assume that $E[S^*]$ follows a geometric Brownian motion, just like S^* . Consider for example the case where S^* represents the product of two variables, an observable stochastic variable (e.g., price), and an unobservable constant (quantity). The learning action seeks to reveal the true value of the unobservable variable (quantity). Before we introduce our continuous-time model, consider a simple one-period discrete example involving a (European) option to invest that expires next period. We can activate a learning action that will reveal the true value of S^* at time $t = 0$ at a cost; or we can wait until maturity of this real option, and if $E[S^*] > X$ we invest and learn about the true value of S^* ex post, else we abandon the investment opportunity. For expositional simplicity (see Exhibit 1) we assume a discrete set of outcomes: the realized value of S^* will differ from $E[S^*]$ by giving a higher value (an optimistic evaluation), a similar one (a most likely evaluation), or a lower one (a pessimistic evaluation) with given probabilities.

[Enter Exhibit 1 about here.]

If management does not take a learning action before option exercise, information will be revealed ex post, resulting in an exercise value for the option different from the expected one. Option exercise might thus prove ex post sub-optimal, as it might result in negative cash flows if the realization of S^* is below X . Similarly, unexercised might also lead to a loss of value, if the true value of S^* is above X . There in fact exist two learning actions, one at time zero, and one at maturity, which are path-dependent. If learning is implemented at time zero, the second opportunity to learn ceases to exist since information has already been revealed that enables subsequent decisions to be made conditioning on the true information, otherwise decisions are made using expectations of uncertain payoffs.

In the following we introduce our continuous-time model with learning and path-dependency. The option is contingent on the state-variable $S = E[S^*]$ that follows a geometric Brownian motion process. The outcomes of information revelation are drawn from a continuous distribution. In the presence of costly learning, there exists an upper and a lower critical boundary within which it is optimal to exercise the (optional) learning action. Outside this range, it is not optimal to pay a cost to learn. The investment is already either too good to worry about possibly lower realized cash flows, or too bad to invest a considerable amount in order to learn more. If learning were costless we would always want to learn early in order to make more informed investment decisions. But if the learning action is too expensive, it may be better to wait and learn ex post. The trade-off between the (ex ante) value added by the learning actions in the form of more informed conditional decisions and the learning cost determines optimal (timing of) control activation.

In the next section we present a basic modeling of real option valuation with embedded learning actions that allows for an analytic solution. Then we introduce multi-stage learning models where more complicated forms of path-dependency are handled with computationally-intensive numerical methods. The last section concludes.

A Basic (Analytic) Model with Learning Actions

We assume that the underlying asset (project) value, S , subject to i optional (and typically costly) learning controls that reveal information, follows a stochastic process of the form:

$$\frac{dS}{S} = \mathbf{m}dt + \mathbf{s}dZ + \sum_{i=1}^N (k_i dq_i), \quad (1)$$

where \mathbf{m} is the instantaneous expected return (drift) and \mathbf{s} the instantaneous standard deviation, dZ is an increment of a standard Wiener process, and dq_i is a jump counter for managerial activation of control i -- a control (not a random) variable.

Under risk-neutral valuation, the asset value S (e.g., see Constantinides, 1978) follows the process

$$\frac{dS}{S} = (\mathbf{m}^*)dt + \mathbf{s}dZ + \sum_{i=1}^N (k_i dq_i) \quad (1a)$$

where the risk-adjusted drift $\mathbf{m}^* = \mathbf{m} - RP$ equals the real drift minus a risk premium RP (e.g., determined from an intertemporal capital asset pricing model, as in Merton, 1973). We do not need to invoke the replication and continuous-trading arguments of Black and Scholes (1973).

Alternatively, $\mathbf{m}^* = r - \mathbf{d}$, where r is the riskless rate of interest, while the parameter \mathbf{d} represents any form of a “dividend yield” (e.g., in McDonald and Siegel, 1984, \mathbf{d} is a deviation from the equilibrium required rate of return, while in Brennan, 1991, \mathbf{d} is a convenience yield). As in Merton (1976), we assume the jump (control) risk to be diversifiable (and hence not priced).

For each control i , we assume that the distribution of its size, $1 + k_i$, is log-normal, i.e., $\ln(1 + k_i) \sim \mathbf{N}(\mathbf{g} - .5\mathbf{s}_{Ci}^2, \mathbf{s}_{Ci}^2)$, with $\mathbf{N}(.,.)$ denoting the normal density function with mean $\mathbf{g} - .5\mathbf{s}_{Ci}^2$ and variance \mathbf{s}_{Ci}^2 , and $E[k_i] \equiv \bar{k}_i = \exp(\mathbf{g}) - 1$. The control outcome is assumed independent of the Brownian motion -- although in a more general setting it can be dependent on time and/or the value of S . Practically we can assume any plausible form. Stochastic differential equation (1a) can alternatively be expressed in integral form as:

$$\ln[S(T)] - \ln[S(0)] = \int_0^T [r - \mathbf{d} - .5\mathbf{s}^2] dt + \int_0^T \mathbf{s}dZ(t) + \sum_{i=1}^N dq_i \ln(1 + k_i). \quad (2)$$

Given our assumptions and conditional on control activation by management,

$$[S^* \mid \text{activation of control } i] = E[S^*](1 + k_i) = S(1 + k_i),$$

making the results from the control action random, and

$$E[S^* \mid \text{activation of control } i] = E[S^*](1 + \bar{k}_i) = S(1 + \bar{k}_i).$$

In the special case of a pure learning control (with zero expected change in value, so $\bar{k}_i = 0$)

$$E[S^* \mid \text{activation of control } i] = S.$$

Useful insights can be gained if we examine the following (simple) path-dependency. Suppose that a single learning control can be activated either at time $t = 0$ at a cost C or at time T (the option maturity) without any (extra) cost -- beyond the exercise price X of the option. The controlled claim (investment opportunity value) F must satisfy the following optimization problem:

$$\text{Maximize}[F(t, S, C)] \quad (3)$$

subject to:

$$\frac{dS}{S} = (r - \mathbf{d})dt + \mathbf{s}dZ + kdq$$

and

$\ln(1 + k)$ is normally distributed with mean: $\mathbf{g} - .5\mathbf{s}_C^2$, and variance: \mathbf{s}_C^2 ,

$$E[k] \equiv \bar{k} = \exp(\mathbf{g}) - 1.$$

Assuming independence between the control and the increment dZ of the standard Wiener process, the *conditional* solution to the European call option is given by:

$$c(S, X, T, \mathbf{s}, \mathbf{d}, r; \mathbf{g}, \mathbf{s}_C) = e^{-rT} E[\max(S^*_T - X, 0) | \text{activation of the control}]. \quad (4)$$

The *conditional* risk-neutral expectation $E[\cdot]$ (derived along the lines of the Black-Scholes model, but conditional on *activation* of a single control at $t = 0$) is:

$$E[\max(S^*_T - X, 0) | \text{activation of the control at } t = 0] = S e^{[(r-d)T + \mathbf{g}]} N(d_{1n}) - X N(d_{2n}) \quad (5)$$

where

$$d_{1n} \equiv \frac{\ln(S/X) + (r - \mathbf{d})T + \mathbf{g} + .5\mathbf{s}^2T + .5\mathbf{s}_C^2}{[\mathbf{s}^2T + \mathbf{s}_C^2]^{1/2}}$$

and

$$d_{2n} \equiv d_{1n} - [\mathbf{s}^2T + \mathbf{s}_C^2]^{1/2}$$

where $N(d)$ denotes the cumulative standard normal density evaluated at d . The value of a *conditional* European put option can be similarly shown. The value of this option conditional on control activation at $t = T$ is the same as the unconditional Black-Scholes European option, since at maturity the option is exercised according to the estimated value $S = E[S^*]$.

Given the rather simple structure we have imposed so far (a single learning action to be activated at either $t = 0$ or at T), the (optimal) value of this real option is

$$\begin{aligned} & \text{Max}[\text{Conditional Value (Learning Activation at } t = 0) - C, \\ & \text{Unconditional Value (Costless learning at } t = T)]. \end{aligned} \tag{6}$$

Numerical Results and Discussion

Table 1 shows the results and accuracy of this analytic model. For comparison purposes, we provide results of a standard numerical (binomial lattice) scheme with $N = 200$ steps. Assuming a costless learning control ($C = 0$ and \bar{k}) we compare real option values for in-the-money, at-the-money, and out-of-the-money options.

[enter Table 1 about here]

If learning is costless, control is always exercised at $t = 0$. The extent of learning potential (captured through the value of s_C) is a very significant determinant of option value. Real options with embedded learning actions are far more valuable than options without any learning potential ($s_C = 0$).

Exhibit 2 illustrates intuition with costly learning. In general there exist an upper S_H and a lower S_L critical asset threshold defining a zone within which it pays to activate the learning action.

[enter Exhibit 2 about here]

Table 2 presents the lower and upper critical asset (project) value thresholds for various values of the learning cost, time to maturity, and learning control volatility. Lower volatility resulting from activation of the learning action implies less uncertainty about the true outcome and has the effect

of narrowing the range when it is optimal to pay a cost to learn. Similarly, increasing learning cost decreases this range, and beyond a point it eliminates it altogether, rendering activation of the learning action a sub-optimal choice.

[Enter Table 2 here]

Multi-Stage Learning

In the previous section we discussed a model (special case) with an analytic solution -- being a function of elements isomorphic to the standard Black and Scholes model. This was possible since learning about the underlying (European) option could occur either at $t = 0$ or (ex post) at $t = T$. With more general assumptions about learning, for example when we can also learn at intermediate times in-between zero and T , or when alternative sequences of learning actions exist that are described by different sets of probability distributions, an analytic solution may in general not be feasible. Two complications arise. One is that numerical methods are needed. The other is that (costly) activation of learning actions induces path-dependency, which should explicitly be taken into account. Martzoukos (1998) assumed independent controls so that path-dependency did not need to be explicitly taken into account. In the following we implement a lattice-based recursive *forward-backward* looking numerical method in order to solve the more general optimization problem

$$\underset{i,t(i)}{\text{Maximize}}[F(t, S, C_i)] \tag{7}$$

subject to:

$$\frac{dS}{S} = (r - \mathbf{d})dt + \mathbf{s}dZ + \sum_{i=1}^N (k_i dq_i)$$

and

$\ln(1 + k_i)$ is normally distributed with mean: $\mathbf{g} - .5\mathbf{s}_{Ci}^2$, and variance: \mathbf{s}_{Ci}^2 ,

$$E[k_i] \equiv \bar{k}_i = \exp(\mathbf{g}) - 1.$$

In the above, adequate conditions must be provided so that the exact control-induced dependency between stages is properly specified.

The general optimization problem in (7) must be solved numerically. Consider an investment option with time-to-maturity T solved on a lattice scheme with N steps of $\Delta t = T/N$ length. In the previous section we had a single decision-node at $t = 0$, since learning at $t = T$ (if information was not completely revealed earlier) would occur without any further action. In this section we consider multi-stage problems, where decision-nodes appear several times ($N_S = 1 - 4$ in our examples) before T . At any of these nodes, learning actions can be activated. In order to determine the optimal activation of these learning controls, their exact interrelation must be specified, which actually determines the problem under consideration. Problems of this type are inherently path-dependent. Activation of learning actions (often at a cost) is conceptually similar to the hysteresis-inducing costly switching of modes of operation treated in Brennan and Schwartz (1985) and Dixit (1989). The main difference is that we allow for a discrete number (instead of a continuum) of actions, at predetermined points in time. The structure is flexible enough to allow for early exercise at any of these nodes (semi-American or Bermudan feature). Between stages the valuation lattice is drawn on the unconditional volatility

$$[\mathbf{s}^2 \Delta t]^{1/2}$$

if no learning has been activated, and on the *conditional* volatility

$$\left[\left(\mathbf{s}^2 \frac{T}{N_s} + \sum_i \mathbf{s}_{C_i}^2 \right) \frac{N_s}{N} \right]^{1/2}$$

if such i learning actions have been activated, with $\frac{N}{N_s}$ being the number of lattice steps per stage. Path-dependency requires that all combinations of sub-problems be analyzed, so that each combination of sub-lattices is distinctly created and used for the pricing of the option. This is achieved through recursive forward-backward looking implementation of the lattice. Option pricing in this context is similar to a discrete optimization problem where the optimum is found through exhaustive search.

In the following we distinguish between fully-revealing actions, where all uncertainty is resolved, and partly-revealing actions, where only part of the uncertainty can be resolved at a time. In the latter case we define the *informativeness* of the learning control to be the percent of total uncertainty that is resolved by a single partly-revealing action. Problems that can be solved with this numerical methodology include the following: A) The single learning action is permissible not only at $t = 0$, but at several discrete intervals before option maturity -- effectively we solve for the optimal timing of the learning action. B) The single learning action can be activated in its entirety, or sequentially in partly-revealing actions. Very likely, such actions have a different cost structure than the single fully-revealing action. To solve this option problem we effectively optimize across two attributes: (a) we solve for the optimal sequence of partial-learning actions, while at the same time determining whether it is optimal to activate partial learning actions; or (b) we exercise a single fully-revealing action (in any of the stages where this is permissible). C) There are several mutually exclusive alternatives of sequences of partly-revealing actions (potentially including the fully-revealing one as a special case) with different cost structures. D) If learning is very costly, we can instead consider only single partly-revealing

mutually exclusive alternatives (instead of a sequence). The remaining uncertainty will be resolved ex post. Effectively we must determine the optimal trade-off between the magnitudes of (partial) learning and their cost, most likely including the fully-revealing alternative (if one exists) in the admissible set of actions. If several stages are involved, we also solve for the optimal timing of the best alternative. In this type of problem we can consider either a continuum or a discrete set of alternative actions. If these actions can only be activated at $t = 0$, an analytic solution is feasible, as in the previous section. E) Other actions with more complicated forms of path-dependency can be included, like different sequences of learning actions (with subsets of actions of varying informativeness and cost structures, etc.).

[Enter Tables 3A and 3B about here.]

In Tables 3A and 3B we provide numerical results for the Multi-Stage option. The case with zero periods ($N_S = 0$) of learning implies that learning can only occur ex post. Cases with one, two, or four periods (stages) can involve active learning at ($t = 0$), at ($t = 0, t = T/2$), at ($t = 0, t = T/3, t = T/2, t = 2T/3$), and of course ex post if information remains to be revealed. In Table 3A we allow for optimal timing of a single fully-revealing and costly action. Optimal timing enhances flexibility and option value as more stages are added (and extrapolation methods like Richardson extrapolation can approximate the continuous limit, as in Geske, 1977). In Table 3B we observe similar results when instead of a single fully-revealing learning action we allow for two (identical) partly-revealing ones. Each has 50% informativeness (and one half the cost) so that if both are activated the learning effectiveness (and total cost) are the same with the base-case of a single fully-revealing action. First we only permit activation of one partly-revealing action at a time. Then (figures in parenthesis) we permit activation of both partial learning actions simultaneously. This is equivalent to optimization for the best of two mutually exclusive

alternatives, the single fully-revealing action or the sequence of two partly-revealing ones (with optimal timing in both cases). In all cases more flexibility can add considerable value.

[enter Table 4 about here]

In Table 4 we illustrate the trade-offs when mutually exclusive learning actions exist with varying levels of informativeness and cost. We assume that $s_C = 0.50$ represents the fully-revealing action. In the first column we present the real option values with zero learning costs ($C = 0/0/0/0$). In the second column, increasing the level of informativeness is at a decreasing marginal cost (economies of scale in information acquisition). In the third column, increasing the level of informativeness is at an increasing marginal cost (diseconomies of scale in information acquisition). In the last column, at first there are economies of scale and then diseconomies of scale. For simplicity, all actions are at time zero only, with any remaining uncertainty to be resolved ex post. We optimize across alternative actions considering a discrete set of mutually exclusive alternatives using the analytic model. Asterisked values represent the optimal choice, which clearly enhances investment (option) value. The reported results can easily be extended to the multi-stage setting with optimal timing of the best action.

In Table 5 we present the numerical results for the important case of a *growth* (compound-type) investment option with learning (Kemna, 1993). The method is a more sophisticated version of the example presented in the classic textbook by Brealey and Myers (2000). The decision-maker can invest C in a first (pilot) investment and get the benefits S_C , plus an option to make the larger scale investment on S by paying capital cost X . Like before, the first investment will improve information about S . To keep a low level of dimensionality, we assume that S_C is a constant fraction of S . If the pilot investment is not undertaken before T , it can be undertaken at T together with the final investment. At time $t = 0$, and without regard to its

learning potential, the pilot project is an investment option (exactly at-the-money), with $S_C = C$, and so is the option on the larger scale investment, with $S = X$. If we account explicitly for learning, the potential for information acquisition and for optimal timing of learning increases investment value tremendously (from a range of 5.145 – 6.368 when $s_C = 0.00$, to a range of 18.573 – 18.616 when $s_C = 0.50$).

[Insert Table 5 about here.]

Conclusions

The present paper addressed and resolved a real options paradox, namely why would management want to resolve (reduce) uncertainty of real (investment) opportunities, given that options in general are non-decreasing functions of volatility. By using a real options framework with incomplete information and costly learning actions that induce path-dependency, we showed that timing of information acquisition is essential. Optimal timing leads to reduction of the cost of potential mistakes and maximizes the value of investment opportunities. This is achieved through improving management's quality of information that leads to superior decision-making. Since information acquisition is in general costly, management effectively seeks the optimal trade-off between the quality of learning, and the cost of learning. Optimal timing of learning must take this into account.

Learning actions can be treated as controlled jumps of random size whose realization is a random variable with a probability distribution known from past data, or elicited through expert opinions (for a Bayesian approach with elicitation and combination of subjective expert opinions, see Lindley and Singpurwalla, 1986, and Cooke, 1991).

The paper discussed various cases of the problem formulation, utilizing an analytic solution as well as a computationally intensive (forward-backward looking) numerical method to solve for the general problem with path-dependency. The analytic solution holds for the case of a single learning action that can be taken only at time zero. The results reveal that costly learning will occur for options that are neither deep out-of-the-money, nor deep in-the-money. Activation of learning actions is optimal within a range defined between two critical thresholds.

Complex multi-stage problems were discussed where the path-dependency induced by the exercise of costly learning actions was treated with a computationally intensive numerical approach. The general problem formulation allows the solution of realistic problems where not only optimal timing of a single learning action is considered, but also the optimal choice of alternative actions with different degrees of learning effectiveness and flexibility. The numerical results clearly demonstrate the value added of learning actions and of flexibility in learning. Since learning actions constitute an inherent part of the managerial toolkit, the classical real options approach that neglects their presence underestimates the value of real options and can lead to inferior decision-making.

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Table 1**Value of Real Option with Costless Learning (Numerical/Analytic)**

		$S = 75.00$	$S = 100.00$	$S = 125.00$
$T = 1$	$\sigma_C = 0.00$	0.005/0.005	3.789/3.793	23.828/23.828
	$\sigma_C = 0.10$	0.090/0.090	5.356/5.362	24.147/24.148
	$\sigma_C = 0.20$	0.856/0.859	8.457/8.468	25.755/25.762
	$\sigma_C = 0.50$	7.361/7.379	19.119/19.142	35.392/35.402
$T = 2$	$\sigma_C = 0.00$	0.085/0.086	5.094/5.101	22.969/22.971
	$\sigma_C = 0.10$	0.271/0.271	6.237/6.245	23.434/23.437
	$\sigma_C = 0.20$	1.130/1.129	8.809/8.820	25.054/25.049
	$\sigma_C = 0.50$	7.262/7.271	18.525/18.548	34.002/34.020
$T = 5$	$\sigma_C = 0.00$	0.701/0.704	6.924/6.933	21.087/21.092
	$\sigma_C = 0.10$	0.972/0.972	7.582/7.592	21.564/21.560
	$\sigma_C = 0.20$	1.810/1.809	9.274/9.286	22.914/22.911
	$\sigma_C = 0.50$	6.897/6.889	16.786/16.807	30.140/30.145

Note: For the underlying asset $S = 100.0$, $r = \mathbf{d} = 0.05$, and $\mathbf{s} = 0.10$, for the option $X = 100.0$, and for the single learning control $\bar{k} = 0.0$, and cost $C = 0.00$. The numerical (lattice) implementation uses $N = 200$ steps.

Table 2**Critical Asset Values (Upper/Lower) with Costly Learning Control**

	Learning cost C (at $t = 0$)				
	0.2	0.5	1.0	2.0	5.0
$T = 1.0$					
$\sigma_C = 0.10$	128.946	120.865	112.842	-----	-----
	78.750	83.971	89.905	-----	-----
$\sigma_C = 0.20$	157.980	145.316	135.287	124.438	-----
	65.688	71.193	76.230	82.564	-----
$\sigma_C = 0.50$	350.681	294.349	254.142	251.813	167.611
	34.926	41.126	47.051	54.472	67.683
$T = 2.0$					
$\sigma_C = 0.10$	135.914	123.947	109.895	-----	-----
	75.445	82.703	93.257	-----	-----
$\sigma_C = 0.20$	165.939	151.287	139.451	125.861	-----
	63.032	68.937	74.603	82.443	-----
$\sigma_C = 0.50$	357.822	299.113	257.334	217.616	167.834
	34.484	40.749	46.757	54.306	67.839
$T = 5.0$					
$\sigma_C = 0.10$	147.718	122.544	-----	-----	-----
	71.521	86.202	-----	-----	-----
$\sigma_C = 0.20$	186.363	164.455	145.806	120.369	-----
	57.660	65.226	73.467	88.872	-----
$\sigma_C = 0.50$	378.416	312.475	265.955	222.055	166.695
	33.336	39.803	46.076	54.096	69.638

Note: For the underlying asset $S = 100.0$, $r = d = 0.05$, and $s = 0.10$, for the option $X = 100.0$, and for the single learning control $\bar{k} = 0.0$. Critical values are calculated using a Newton-Raphson scheme and the analytic model.

Table 3A**Multi-Stage Real Option with Learning (Numerical)**

<i>Optimal Timing of a Single Fully-Revealing Learning Action</i>					
		$C = 0.50$	$C = 1.00$	$C = 2.00$	$C = 5.00$
$\sigma_C = 0.00$	$N_S = 0$	5.094	5.094	5.094	5.094
$\sigma_C = 0.10$	$N_S = 1$	5.737	5.237	5.094	5.094
	$N_S = 2$	5.784	5.381	5.094	5.094
	$N_S = 4$	5.831	5.495	5.096	5.094
$\sigma_C = 0.20$	$N_S = 1$	8.309	7.809	6.809	5.094
	$N_S = 2$	8.349	7.875	6.942	5.094
	$N_S = 4$	8.361	7.905	7.041	5.205
$\sigma_C = 0.50$	$N_S = 1$	18.025	17.525	16.525	13.525
	$N_S = 2$	18.084	17.608	16.657	13.804
	$N_S = 4$	18.108	17.644	16.716	13.934

Note: For the underlying asset $S = 100.0$, $r = \mathbf{d} = 0.05$, and $\mathbf{s} = 0.10$, for the option $X = 100.0$, and $T = 2.00$, and for the learning controls $\bar{k} = 0.0$. For the numerical (lattice) we used $N = 200$ steps.

Table 3B

Multi-Stage Real Option with Costly Learning (Numerical)

<i>Optimal Timing of a Sequence of Two Partly-Revealing Learning Actions</i>					
One permitted at a time (both permitted at a time)					
		$C1 = C2 = 0.25$	$C1 = C2 = 0.50$	$C1 = C2 = 1.00$	$C1 = C2 = 2.50$
$\sigma_{C1} = \sigma_{C2} = 0$	$N_S = 0$	5.094	5.094	5.094	5.094
$\sigma_{C1} = 0.0707$	$N_S = 1$	(5.737) 5.444	(5.237) 5.194	(5.094) 5.094	(5.094) 5.094
	$N_S = 2$	(5.784) 5.768	(5.381) 5.341	(5.094) 5.094	(5.094) 5.094
$\sigma_{C2} = 0.0707$	$N_S = 4$	(5.836) 5.827	(5.501) 5.478	(5.123) 5.123	(5.094) 5.094
$\sigma_{C1} = 0.1414$	$N_S = 1$	(8.309) 6.948	(7.809) 6.698	(6.809) 6.198	(5.094) 5.094
	$N_S = 2$	(8.349) 8.331	(7.875) 7.870	(6.995) 6.995	(5.181) 5.181
$\sigma_{C2} = 0.1414$	$N_S = 4$	(8.377) 8.375	(7.949) 7.949	(7.152) 7.149	(5.389) 5.389
$\sigma_{C1} = 0.3535$	$N_S = 1$	(18.025) 13.394	(17.525) 13.144	(16.525) 12.644	(13.525) 11.144
	$N_S = 2$	(18.084) 18.041	(17.608) 17.570	(16.657) 16.650	(14.050) 14.050
$\sigma_{C2} = 0.3535$	$N_S = 4$	(18.114) 18.095	(17.656) 17.644	(16.769) 16.766	(14.287) 14.287

Note: For the underlying asset $S = 100.0$, $r = d = 0.05$, and $s = 0.10$, for the option $X = 100.0$, and $T = 2.00$, and for the learning controls $\bar{k} = 0.0$. For the numerical (lattice) we used $N = 200$ steps.

For the case of partly-revealing actions, the combined standard deviation if both are activated equals 0.10 in the first, 0.20 in the second, and 0.50 in the third case, so that the combined result is equivalent to that of a fully-revealing action for comparison purposes with a single action. In parenthesis we provide values when we permit both partly-revealing actions to also occur simultaneously (equivalent to fully-revealing).

Table 4

**Mutually Exclusive Alternative Learning Actions with Informativeness
Cost-benefit Trade-offs and Scale (Diss)economies**

$S = 100$				
$X = 100$	$C = 0/0/0/0$	$C = 0/1.5/2/3$	$C = 0/1/4/15$	$C = 0/2/3/15$
$\sigma_c = 0.00$	5.101	5.101	5.101	5.101
$\sigma_c = 0.10$	6.245	4.745	*5.245	4.245
$\sigma_c = 0.20$	8.820	6.820	4.820	*5.820
$\sigma_c = 0.50$	*18.548	*15.548	3.548	3.548
$S = 125$				
$X = 100$	$C = 0/0/0/0$	$C = 0/1.5/2/3$	$C = 0/0.4/2.5/12$	$C = 0/1.5/2/12$
$\sigma_c = 0.00$	22.971	22.971	22.971	22.971
$\sigma_c = 0.10$	23.437	21.937	*23.037	21.937
$\sigma_c = 0.20$	25.049	23.049	22.549	*23.049
$\sigma_c = 0.50$	*34.020	*31.020	22.020	22.020

Note: For the underlying asset $S = 100.0$, $r = \delta = 0.05$, and $\sigma = 0.10$, for the option $X = 100.0$, and $T = 2.00$, and for the learning controls $\bar{k} = 0.0$. The fully-revealing action is defined by $s_c = 0.50$. All other controls have partial only revealing potential. Asterisk values represent the optimal choice.

Table 5

Multi-Stage Real Option with Growth and Learning (Numerical)

Optimal Timing of Growth Investment with Fully-Revealing Learning Potential

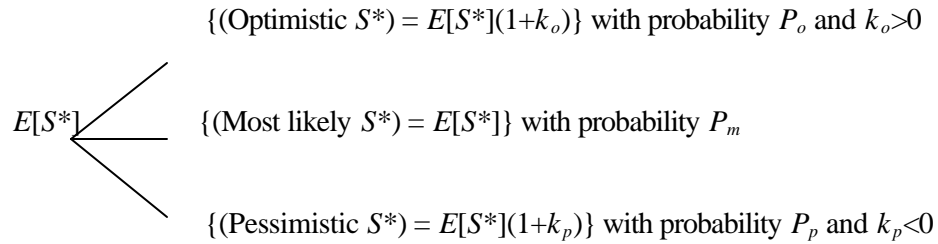
		$C = 1.0,$ $S_C = (1\%)S$	$C = 5.0,$ $S_C = (5\%)S$	$C = 10.0,$ $S_C = (10\%)S$	$C = 25.0,$ $S_C = (25\%)S$
$\sigma_C = 0.00$	$N_S = 0$	5.145	5.349	5.604	6.368
$\sigma_C = 0.10$	$N_S = 1$	6.237	6.237	6.237	6.368
	$N_S = 2$	6.248	6.280	6.371	6.852
	$N_S = 4$	6.256	6.333	6.478	7.039
$\sigma_C = 0.20$	$N_S = 1$	8.809	8.809	8.809	8.809
	$N_S = 2$	8.825	8.827	8.842	9.009
	$N_S = 4$	8.825	11.100	8.897	9.210
$\sigma_C = 0.50$	$N_S = 1$	18.525	18.525	18.525	18.525
	$N_S = 2$	18.560	18.560	18.560	18.561
	$N_S = 4$	18.573	26.313	18.587	18.618

Note: For the underlying asset $S = 100.0$, $r = \mathbf{d} = 0.05$, and $\mathbf{s} = 0.10$, for the option $X = 100.0$, and $T = 2.00$, and for the learning controls $\bar{k} = 0.0$. For the numerical (lattice) we used $N = 200$ steps.

Exhibit 1

Information Revelation and the Cost of a Mistake

Information Revelation (at $t = 0$, or $t = T$)



Potential Cost of a Mistake with Ex Post Learning ($t = T$)

Real Option is Exercised if $E[S^*] > X$, but:

If (Pessimistic S^*) $< X$, there is Probability P_p of a Realized Loss = (Pessimistic S^*) $- X$

Real Option is not Exercised if $E[S^*] \leq X$, but:

If (Optimistic S^*) $> X$, there is Probability P_o of an Opportunity Cost = (Optimistic S^*) $- X$

Exhibit 2

Optimal Activation and Optimal Timing of Costly Learning

at $t = 0$		at $t = T$		
CRITICAL ASSET VALUE	OPTIMAL DECISION	CRITICAL ASSET VALUE	OPTIMAL DECISION	
$S_H < S$	} NO ACTIVATION OF LEARNING	} $X < S$	} INVEST (w/ ACTIVATION if not activated before)	
$S_L \leq S \leq S_H$				} ACTIVATION
$S < S_L$				
		$S \leq X$	ABANDON	

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